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TREATING EQUALS EQUALLY AND UNEQUALS UNEQUALLY IN ONE-SIDED MATCHING MARKETS

TRATO DIFERENCIADO POR TIPO EN MERCADOS DE ASIGNACIÓN UNILATERALES

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Resumen:
En el mercado de asignación unilateral de Shapley y Scarf (1974) introducimos el axioma de "trato diferenciado por tipo" (TEEUU, por sus siglas en inglés). Modelamos el mercado de asignación como una función medible que asigna a cada tipo de agente un tipo de bien. Establecemos: 1) un método, fundamentado en la literatura de transporte óptimo, que permite encontrar una asignación en el núcleo que cumpla con el axioma TEEUU siempre que exista (el método consiste en buscar una asignación Pareto óptima); 2) condiciones que garantizan la existencia de una asignación en el núcleo que cumpla con TEEUU.

Abstract:
We introduce in the one-sided assignment game by Shapley and Scarf (1974) the requirement called "Treating equals equally and unequal unequally" (TEEUU). We model an assignment as a measurable function that assigns a type of good to each type of agent. We establish: 1) a method, originated in optimal transportation theory, to find a TEEUU assignment in the core whenever it exists- by searching a Pareto optimal assignment- and 2) conditions under which a TEEUU assignment in the core always exists.

Clasificación JEL/JEL Classification: C61, C62, C78, D51, D79

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1. Introduction

We introduce a normative criterion, "Treating equals equally and unequal unequally" (TEEUU), and study the existence of a TEEUU assignment in a one-sided[1] matching market that belongs to the core of the economy.

An assignment is TEEUU if agents of the same type are assigned to goods of the same type. Subsidized housing is a one-sided problem where TEEUU might apply. There, a type for a family is the number of children, and a type for a subsidized apartment is the number of rooms; in this context, a TEEUU assignment is, for instance, one that assigns apartments with a large number of bedrooms to large families.

The criteria also apply to two-sided matching markets; however, we do not study such markets in this work. In the school choice problem (Abdulkadiroglu and Sönmez, 2003), for instance, a matching is fair if, whenever a student is rejected by a given school, all students accepted at that school outperformed that student on the priority order. In this approach, all students of the same type are treated equally, and fair matchings belong to the core. Another approach to fairness considers that, moreover, less performing students should have access to schools that provide further support and help to improve their performance; thus, unequals should be treated unequally. We investigate in this paper the normative criteria "Treating equals equally and unequals unequally" in a one-sided matching market; we call such assignments TEEUU matchings. At these matchings, students of the same type are assigned to school of the same type, thus a matching is TEEUU if and only if it is in the core and students of the same type are assigned to schools of the same type, i.e., type-exclusive.

Unsurprisingly, core TEEUU matchings do not always exist. The paper deals with the following two questions: can one find a TEEUU matching in the core whenever it exists? Are there conditions that guarantee the existence of a TEEUU matching in the core?

We model the concept of assignment as a measurable function that assigns a type of good to each type of agent. We use tools developed in the literature of transportation theory to carry out our analysis. Our main result establishes that if an assignment is the solution of a Pareto optimality problem, specified as a maximization

[1] We follow the definition by Echenique *et al.* (2023: 37): "A one-sided matching allocation problem consists of a set A of n agents, a set G of n indivisible goods, and preferences of agents over goods".

problem -not necessarily linear- then it is in the core of the economy. We establish specific continuity, differentiability, and topological conditions for the non-emptiness of the core.

Garg *et al.* (2021) study a one-sided matching problem and define equal-type envy-free assignment, which requires eliminating envy-freeness for agents of the same type. TEEUU differs from equal-type envy-freeness for dealing with situations where one or many component(s) of the good, interpreted as its type, reflects a need and, thus, requires a level of care that differs from type to type. Therefore, TEEUU covers equal-type envy-freeness and requires agents from different types to get different types of objects.

Following Aumann (1964), our work belongs to the strand of the literature that studies the existence of assignments in the core and approximations of the core in variants of the Assignment game with continuous types, or population of agents, like Kaneko and Wooders (1985) and Kovalenkov and Wooders (2003). Our paper is closer to Gretsky *et al.* (1992), who establish, in particular, the equivalence of core solutions and Walrasian equilibrium. Our main result, Theorem 4, demonstrates that a TEEUU solution to the Pareto optimality problem- not necessarily linear- belongs to the core of the economy. The result is key since it allows us to rest on Levin (2004) and Carlier (2003) to establish our existence conditions. Since the TEEUU requirement is new, our results are independent of classical existence results in the literature.

The remainder of the paper is organized as follows. Section 2 presents the type-exclusive assignment economy in which the assignment concept is a measurable function. Theorem 4 establishes a relation between the core of type-exclusive assignment economy and an optimal transport problem. Finally, we propose a model where there is a non-atomic measure on the set of types of goods and agents and other conditions to ensure the non-emptiness of the core in the type-exclusive assignment economy. We conclude with some general comments on possible extensions. The Appendix presents an extension of Debreu's preferences representation theorem.

2. Type-exclusive assignment model

We require assignments to be such that two or more agents of the same type are assigned to goods of the same type, and vice versa, a normative criterion potentially incompatible with the existence of a core stable assignment.

2.1 *Type-exclusive assignment economy*

Consider an economy that has a population of agents and a population of indivisible goods. Each indivisible good is labeled with a single type, g, which describes the different characteristics that fully characterize a good. The set of types of goods is G. Each agent is labeled with a type, a, where a particular type represents the preference relation $\precsim_a$ over G. The set of types of agents is A. Assume that A and G are compact Borel spaces; that is, they are separable and compact metric spaces.

We assume that preference relations $\{\precsim_a\}_{a\in A}$ are represented by a continuous utility function $u : A \times G \to [0,1]$, that is:

$$g \precsim_a g' \; if \; and \; only \; if \; u(a,g) \leq u(a,\ g'). \tag{1}$$

Two comments are in order about the utility function: 1) continuity is a requirement one cannot dispense of for our results to hold (the definition of continuity should be understood in the context of general topology), and 2) the utility function $u(\cdot,\cdot)$ represents the preference relations of all types of agents, so when one considers transformations f of $u(\cdot,\cdot)$ that also represent the preference relations of all types, they might treat arguments a and g differently. Specifically, since no cardinality of utility is assumed for our purpose, any transformation $u^f(\cdot,\cdot) = f(u(\cdot,\cdot))$ that is strictly increasing also represents the original preference relations, that is, if $g \precsim_a g'$, then:

$$u^f(a,g)) \leq u^f(a,g')) \; if \; and \; only \; if \; u(a,g) \leq u(a,g') \tag{2}$$

$$for \; all \; a \in A.$$

This is the only monotonicity restriction required on f.[2]

Let $B(A)$ and $B(G)$ be the Borel σ-algebras of A and G, respectively. Probability measures η and ν assign a population distribution over the sets A and G, respectively. Finally, we denote the population of agents and the population of indivisible goods by the probability measure spaces:

[2] For conditions in $\{\precsim_a\}_{a\in A}$ that guarantee the existence of $u(\cdot,\cdot)$, see, for example, Levin (1983); Rachev and Ruchendorf (1998, Theorem 5.5.18, p. 337); or Bridges and Mehta (2013, Theorem 8.2.6, p. 146) (see Appendix).

$$\mathbf{A} := (A,\ \mathcal{B}(A), \eta) \tag{3}$$

and

$$\mathbf{G} := (G, \mathcal{B}(G), \nu), \tag{4}$$

respectively. A *type-exclusive* assignment for the economy $\mathcal{E}$ is a measurable function $u : A \to G$. A type-exclusive assignment μ for $\mathcal{E}$ is TEEUU if for each set of types of indivisible goods E in $B(G)$, the amount $\nu(E)$ of indivisible goods is proportional to the amount $\eta\left(\mu^{-1}(E)\right)$ of agents. In other words, a type-exclusive assignment μ for $\mathcal{E}$ is TEEUU if:

$$\eta\left(\mu^{-1}(E)\right) = \nu(E) \qquad \textit{for all } E \in \mathcal{B}(G). \tag{5}$$

A *type-exclusive assignment economy* is a quadruple $\mathcal{E} := (\mathbf{A},\ \mathbf{G},\ u,\ \mu_0)$, where $\mathbf{A}$ is a population of agents as in (3), $\mathbf{G}$ is a population of indivisible goods as in (4), u is a continuous function that satisfies (1), and, finally, $\mu_0 : A \to G \cup \emptyset$ is a measurable function which assigns for each type of agent a in A the agent's initial endowment $\mu_0(a)$ in G and satisfies (5), or all agents have the empty set $\emptyset$ as initial endowment. The function μ_0 is called the initial *type-exclusive endowment*.

The following are two examples of type-exclusive economies and assignments.

Example 1: Consider an economy $\mathcal{E}$ where A and G are finite sets with the same cardinality n and the function u in (1) is represented as a square matrix $[u(a,g)]_{\{a\in A,\ g\in G\}}$ of rank n. Let η and ν be uniform probability distributions over the sets A and G, respectively; that is, $\eta(a) = \nu(b) = \frac{1}{n}$ for all $a \in A$ and $g \in G$. In this case, any bijective function $\mu : A \to G$ is a feasible type-exclusive assignment.

Remark: The economy in Example 1 is a particular case of Shapley and Scarf (1974).

Example 2: Consider an economy $\mathcal{E}$ where A and G are both the interval $[0,1]$ and the function u in (1) is $u(a,g) = a^2 + g^2$. Let η and ν be uniform probability distributions over the sets A and G, respectively; that is, $\eta(da) = \nu(dg) = 1$ for all $a \in A$ and $g \in G$, where $\eta(da)$ and $\nu(dg)$ are density functions. In this case, the functions $\mu_1(a) = a$ and $\mu_2(a) = 1 - a$ are TEEUU type-exclusive assignments for $\mathcal{E}$.

We define now the core for this economy.

Definition 3: The *core*$\mathbf{C}(\mathcal{E})$ of an economy $\mathcal{E}$ is the set of all TEEUU type-exclusive assignment μ such that there is no coalition $S \in \mathcal{B}(A)$ with $(\nu(S) > 0)$ and type-exclusive assignment γ that satisfies the following three conditions:

- *E1. i)* $\eta(\gamma^{-1}(E)) = \nu(E)$ for all $E \in \mathcal{B}(G) \cap \overline{\gamma(S)}$ where $\overline{\gamma(S)}$ is the topological closure of $\gamma(S)$;
  ii) $\overline{\mu(S)} = \overline{\gamma(S)}$, where $\overline{\mu(S)}$ and $\overline{\gamma(S)}$ refer to the topological closures of $\mu(S)$ and $\gamma(S)$, respectively.
- *E2.* $u(a, \mu(a)) \leq u(a, \gamma(a))$ η-almost everywhere in S.
- *E3.* There exists $D \in \mathcal{B}(A) \cap S$ with $\eta(D) > 0$, and $u(a, \mu(a)) < u(a, \gamma(a))$ η-almost everywhere in D.

Condition *E1-i)* refers to the feasibility of the assignment; a mass of goods is assigned to a mass of agents in equivalent proportions. Put differently, 30% of the agents cannot be assigned to 65% of goods. Condition *E1-ii)* ensures that the blocking coalition S does not require goods held by agents out of S. Conditions *E2* and *E3* refer to the incentives that individuals in coalition S have to improve with respect to their assignments. In conditions *E2* and *E3* the statement "η-almost everywhere" means that these conditions can fail only in a subset of η-measure zero.

2.2 *Looking for a Pareto optimal assignment*

In this section, we consider a social planner who searches for a type-exclusive assignment that is Pareto optimal (when the initial matching is the empty one, i.e., $\mu_0(a) = \emptyset$ for all a in A). Let $\mathcal{L}$ be the set of all feasible type-exclusive assignment, i.e.,

$$\mathcal{L} := \left\{\mu : A \to G \ : \ \eta\left(\mu^{-1}(E)\right) = \nu(E) \ \ for\ all\ E \in B(G)\right\}. \quad (6)$$

Consider the social planner's problem in an economy $\mathcal{E}$:

$$\max_{\mu \in \mathcal{L}} \int_A u(a, \mu(a))\, \eta(da) \quad (7)$$

with $\mathcal{L}$ as in (6). We observe that $\mathcal{L}$ can be empty, as in Example 6.

If μ^* is a solution to the planner's problem, one cannot strictly increase the utility of a given type without lowering the others; thus μ^* is Pareto optimal. Obviously, it is not the only one, and when u is transformed by a new utility function that satisfies (2), the new solution might be another Pareto optimal assignment. The dependence on the utility representation is not critical for our present purpose, which is to establish the relation between the core of the economy $\mathbf{C}(\mathcal{E})$ and the social planner's problem (7).

Theorem 4: A type-exclusive assignment μ^* is solution to the social planner's problem (7), then μ^* is in $\mathbf{C}(\mathcal{E})$.

Proof. Suppose that μ^* is solution to (7) and it is not in $\mathbf{C}(\mathcal{E})$. Then there exists $S \in \mathcal{B}(A)$ (with $\eta(s) > 0$) and a feasible type-exclusive assignment γ which satisfy *E1*-*E2* in Definition 3.

Now, consider the exclusive assignment:

$$\mu(a) = \begin{cases} \mu^*(a) & \text{if} \quad a \notin S \\ \gamma(a) & \text{if} \quad a \in S. \end{cases}$$

Note that $\overline{\gamma(S)} = \overline{\mu(S)} = \overline{\mu^*(S)}$. Let $E \in \mathcal{B}(G)$. Then by *E1-ii)*, the properties of the inverse image, and (5), we have that:

$$\begin{aligned}
\eta\left(\mu^{-1}(E)\right) &= \eta\left(\mu^{-1}\left(E \cap \overline{\gamma(S)}\right)\right) + \eta\left(\mu^{-1}\left(E \cap \left(G \setminus \overline{\gamma(S)}\right)\right)\right) \\
&= \eta\left(\gamma^{-1}\left(E \cap \overline{\gamma(S)}\right)\right) + \eta\left(\mu^{*-1}\left(E \cap \left(G \setminus \overline{\gamma(S)}\right)\right)\right) \\
&= \eta\left(\gamma^{-1}\left(E \cap \overline{\gamma(S)}\right)\right) + \eta\left(\mu^{*-1}\left(E \setminus E \cap \overline{\gamma(S)}\right)\right) \\
&= \eta\left(\gamma^{-1}\left(E \cap \overline{\gamma(S)}\right)\right) + \eta\left(\mu^{*-1}(E) \setminus \mu^{*-1}\left(E \cap \overline{\gamma(S)}\right)\right) \\
&= \eta\left(\gamma^{-1}\left(E \cap \overline{\gamma(S)}\right)\right) + \eta\left(\mu^{*-1}(E)\right) - \eta\left(\mu^{*-1}\left(E \cap \overline{\gamma(S)}\right)\right) \\
&= \nu\left(E \cap \overline{\gamma(S)}\right) + \nu(E) - \nu\left(E \cap \overline{\gamma(S)}\right) \\
&= \nu(E).
\end{aligned}$$

Thus, μ is a TEEUU exclusive assignment for $\mathcal{E}$. Moreover, by *E2* and *E3*, it satisfies that:

$$\int_A u(a,\ \mu^*(a))\eta(da) = \int_{A-S} u(a,\mu^*(a))\,\eta(da) + \int_S u(a,\mu^*(a))\,\eta(da)$$

$$< \int_{A-S} u(a,\mu^*(a))\,\eta(da) + \int_S u(a,\gamma(a))\,\eta(da)$$

$$= \int_{A-S} u(a,\mu(a))\,\eta(da) + \int_S u(a,\mu(a))\,\eta(da)$$

$$= \int_A u(a,\mu(a))\,\eta(da)\,.$$

Therefore, μ^* is not optimal for (7), which is a contradiction. So, we conclude that μ^* is in the core.

Example 5: Consider an economy $\mathcal{E}$ as in Example 2. In this case, $\mathcal{L}$ (as in (6)) is the set of all bijective functions $\mu : A \to G$. The social planner's problem is given by the optimization problem:

$$\max_{\mu\in\mathcal{L}} \frac{1}{n}\sum_{a\in A} u(a,\mu(a))\,.$$

2.3 *The core and TEEUU allocations*

Consider an economy $\mathcal{E}$. If μ^* is a solution to the problem (7), then by Theorem 4, μ^* is in $\mathbf{C}(\mathcal{E})$ and therefore the core of $\mathcal{E}$ is not empty. Nevertheless, the set of TEEUU allocation $\mathcal{L}$ in (6) is not necessarily compact nor convex; in fact it may be empty (as in Example 6). In any case, (7) may have no solution. The following example provides a case where the set of TEEUU allocations $\mathcal{L}$ is empty.

Example 6: Consider a population of agents $\mathbf{A} := (A, \mathcal{B}(A), \delta_a)$ and a population of goods $\mathbf{G} := (G, \mathcal{B}(G), \nu)$, where δ_a is Dirac probability measure at $a \in A$ and ν is defined by:

$$\nu(E) := \frac{1}{2}\delta_{g_1}(E) + \frac{1}{2}\delta_{g_2}(E) \qquad \forall E \in \mathcal{B}(G)\,,$$

where δ_{g_1} and δ_{g_2} are Dirac probability measures on G with $g_1 \neq g_2$. This example describes a situation in which we only have two types of goods, and one type of agent. In this case $\mathcal{L} = \emptyset$.

Remark 7: Consider an economy as in Example 1, where A and G are finite sets with the same cardinality n, and η and ν are uniform probability distributions over the sets A and G, respectively. In this case, the social planner's problem has a solution; see, for example, Koopmans and Beckmann (1957) or Shapley and Scarf (1974). Moreover, according to Theorem 4, the core of this economy is not empty.

2.4 *Non-atomic sets of types and the non-emptiness of* $\mathbf{C}(\mathcal{E})$

In this section, we establish particular conditions under which if we have non-atomic sets of types, then the core of an economy $\mathcal{E}$ is nonempty. We consider the following assumptions:

- *A.1. Non-atomic sets of types.* The set of types of agents A and the set of types of indivisible goods G are compact subsets of R^n, and η is a probability measure on $B(A)$, which is absolutely continuous with respect to n-dimensional Lebesgue measure.
- *A.2. Heterogeneity on utility.* Let U be a differentiable function in $A \times G$. If $g', g \in supp\,(\nu)$ with $g \neq g'$, where $supp\,(\nu)$ denotes the support of probability measure ν, then:

$$\frac{\partial u}{\partial a}(a, g) \neq \frac{\partial u}{\partial a}(a, g').$$

- *A.3.Convexity/concavity in types of agents.* The set A is convex, and for each $g \in supp\,(\nu)$, the function $a \to u\,(a, g)$ is strictly concave or strictly convex.
- *A.4. Boundedness on the heterogeneity of types of agents.* The set int(supp(η)) is not empty and its complement is Lebesgue negligible;[3] for every $g \in supp(v)$, $a \to (a, g)$ is differentiable and for any $a \in$ supp(η), there exists a neighborhood V of a and a number $c_a > 0$, such that:

$$||u\,(a_1, g) - u\,(a_2, g)||\,| \leq c_a a_1 - a_2 \qquad for\ all\ a_1, a_2 \in V,\ g \in \mathrm{supp}\,(\nu)$$

Proposition 8: Consider assumptions *A1*, *A2*, and *A3*. Then the problem (7) admits at least one solution.

Proof. See Levin (2004), Theorems 1.2 and 1.3.

A few comments on the conditions are in order. *A1* Non-atomic set of types: previous results do not require the assumption; however, it might be imposed from the outset of our model since most

[3] Where int(supp(η)) denotates the interior set of supp(η).

economic models deal with continuous- and even differentiable- distribution functions of types. Without *A1*, building a function that copes with TEEUU is technically difficult. *A2/A3* Heterogeneity on utilities and heterogeneity in the types of agents; in both cases, a lack of heterogeneity leads to the same evaluation of two types of agents/goods by the objective function in the Pareto optimality problem (7), thus leading to the assignment of the same types in the solution, violating TEEUU. *A3* can indeed be relaxed. Nevertheless, the changes in heterogeneity in types of agents should not be too steep, as imposed in *A4*.

Proposition 9: Consider assumptions *A1*, *A2*, and *A4*. Then the problem (7) admits at least one solution.

Proof. See Levin (2004), Theorem 1.4. Carlier (2003) proposes similar conditions of Proposition 9 for metric spaces.

The next example satisfies the assumptions *A1*, *A2*, and *A4*. The reader can find other interesting examples in Levin (2004).

Example 10: Let $\mathcal{E} := (\mathbf{A}, \mathbf{G}, u, \mu_0)$ be a type-exclusive assignment economy, where A and G are convex and compact subsets of $\Re^n$; η is absolutely continuous with respect to the Lebesgue measure on $A; u(a, g) = -\sum (a_i - g_i)^2$ for $a := (a_1, \ldots, a_2)$ and $g := (g_1, \ldots, g_n)$, and μ_0 is any agent's initial endowment. Let $\mu^*(a) = Ha + b$, where H is symmetric and positive semi-definite matrix, and $b \in \Re^n$. If $\nu(E) = \eta\left(\mu^{*-1}(E)\right)$ for all $E \in \mathcal{B}(G)$, then μ^* is the unique optimal solution of (7), and it is in the core of $\mathcal{E}$.

3. Concluding remarks

We introduce the normative requirement that we call TEEUU and develop an approach based on optimal transport theory to find a TEEUU assignment whenever it exists. While TEEUU proves to be a strong requirement, the approach we develop, in contrast, is fruitful in type-exclusive economies. We believe it is indeed a versatile and powerful tool that can be used in general assignment problems, a task that we hope to carry out in future research.

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Appendix: An extension of the Debreu's preferences representation theorem

Consider an economy assuming that A and G are compact Borel spaces; that is, they are complete, separable, and compact metric spaces. For the preference relations $\left\{\precsim_a\right\}_{a\in A}$, we assume that:

- ○ *H.1. rationality:* for each a in A, $\precsim_a$ is a is a complete and transitive order relation;
- ○ *H.2. continuity in the goods:* for each a in A and g' in G, the sets $\left\{g \in G : g' \precsim_a g\right\}$ and $\left\{g \in G : g \precsim_a g\right\}$ are closed;
- ○ *H.3. continuity in the agents:* for any g',$g \in G$ the set $\left\{a \in A : g' \precsim_a g\right\}$ is closed.

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The following theorem constitutes an extension of Debreu's preference representation theorem (Debreu, 1954). The proof is provided in Levin (1983); in Rachev and Rüschendorf (1998, Theorem 5.5.18, p. 337); and in Bridges and Mehta (2013, Theorem 8.3.6, p. 146).

Theorem 11: Let A be the set of type of agents, and G be the set of type of indivisible goods. Assume as in (1) and (3) that A and G are compact metric spaces, and let G, in addition, be separable. Suppose that the preference relations $\left\{\precsim_a\right\}_{a\in A}$ satisfy *H1*, *H2* and *H3*. Then, there exists a continuous function $u : A \times G \to [0,1]$, such that:

$$\forall a \in A, \qquad g \precsim_a g' if \text{ } and \text{ } only \text{ } if \text{ } u\left(a,g\right) \leq u\left(a,g'\right).$$