AN ELEMENTARY TRANSFERS PROCEDURE FOR SHARING THE JOINT SURPLUS IN GAMES WITH EXTERNALITIES

UN PROCEDIMIENTO ELEMENTAL DE TRANSFERENCIAS PARA REPARTIR EL EXCEDENTE CONJUNTO EN JUEGOS CON EXTERNALIDADES

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Resumen: Sánchez-Pérez (2017, Teorema 3) presenta una formulación analítica de todas las soluciones para juegos con externalidades que satisfacen los axiomas de linealidad, simetría y eficiencia. El principal propósito en este artículo es presentar una formulación con una interpretación más intuitiva. En particular, estamos interesados en una interpretación basada en la idea de transferencias entre los jugadores.

Abstract: Sánchez-Pérez (2017, Theorem 3) presents an analytic characterization for all solutions for games with externalities that satisfy the axioms of linearity, symmetry, and efficiency. The main goal of this paper is to recast such formulation to a more intuitive interpretation. In particular, we are interested in an interpretation based on the idea of transfers among players.

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1. Introduction

Economic activities on the macro and micro levels often entail widespread externalities. This leads to disputes regarding the compensation levels for the various parties affected. The problem of how to fairly divide a surplus obtained through cooperation is one of the most fundamental issues studied in coalitional game theory, and it is relevant to a wide range of economic and social situations. These issues are often difficult to resolve, especially in environments with externalities, where the benefits of a group depend not only on its members but also on the arrangement of agents outside the group. This is the general problem to which this paper contributes. In this line, the concept of partition function form games effectively modeled such a problem in Lucas and Thrall (1963). In this sense, a game with externalities assigns a value to each pair consisting of a coalition and a coalition structure. The advantage of this model is that it considers both internal factors (coalition itself) and external factors (coalition structure) that may affect cooperative outcomes and allow to go deeper into cooperation problems. Thus, it is closer to real-life situations but more complex to analyze.

Given a game with externalities, we are usually interested to know how the "fruits" of cooperation are shared among the involved players. A solution for this kind of game is a function that assigns to every game a payoff vector, where each component of the vector is the payoff assigned to the corresponding individual player. Usually, the payoffs assigned to the individual players are based on their contribution to the different coalitions they are or can be members.

There has been a surge of literature that deals with solutions for games with externalities. The first paper that proposed a value concept for this type of games was Myerson (1977). More recently, Albizuri *et al.* (2005), Macho-Stadler *et al.* (2007), Ju (2007), and Pham Do and Norde (2007) apply the axiomatic approach to characterize a value for these games. All of them satisfy three basic properties in the cooperative game theory framework: linearity, symmetry, and efficiency.

Linearity allows us to determine the value of any game that can be established as their linear combination. Symmetry is an elementary property that states that players' names do not matter, and efficiency merely states that the value is a distribution of the value of the grand coalition among these players.

In particular, notice that the value does not always generate a Pareto efficient outcome; it would be Pareto efficient only when forming the grand coalition generates the largest total surplus. Hence, we have in mind economic environments where doing so is the most efficient way of organizing society. International negotiations and many other interesting economic environments clearly satisfy that the players maximize total surplus when they make decisions jointly because they can internalize the externalities.

In this work, we propose a simple mechanism (based on transfers among players) to share the joint surplus of cooperation among them by means of the different externalities to consider. In particular, we show that every linear-symmetric-efficient solution presented in Sánchez-Pérez (2017, Theorem 3) is obtained from this mechanism.

2. Preliminaries

Let $N = \{1, 2, ..., n\}$ be a fixed nonempty finite set, and let the members of N be interpreted as players in some game situation. Given N, let PT be the set of partitions of N, so:

$$\{S_1, S_2, ..., S_m\} \in PT \text{ iff } \bigcup_{i=1}^{m} S_i = N, \;\; S_j \cap S_k = \emptyset \forall j \neq k$$

Also, let $EC = \{(S, Q) \,| S \in Q \in PT\}$ be the set of *embedded coalitions*; that is, the set of coalitions and specifications on how the other players are aligned.

Definition 1: A game with externalities is a mapping

$$w : EC \to \mathbb{R}$$

The set of games with externalities with player set N is denoted by G, i.e.,

$$G = \{w : EC \to \mathbb{R}\}$$

The value $w\,(S, Q)$ represents the payoff of coalition S, given the coalition structure Q forms. In this kind of game, the worth of some coalition depends not only on what the players of such coalition can jointly obtain but also on how the other players are organized. We assume that, in any game situation, the universal coalition N (embedded in $\{N\}$) would actually form, so that the players would have $w\,(N, \{N\})$ to divide among themselves. But we also anticipate that the actual allocation of this worth would depend on all the other potential worths $w(S,Q)$, as they influence the relative bargaining strengths of the players.

Given $w_1, w_2 \in G$ and $c \in \mathbb{R}$, we define the sum $w_1 + w_2$ and the product cw_1, in G, in the usual form, i.e., $(w_1 + w_2)(S,Q) = w_1(S,Q) + w_2(S,Q)$ and $(cw_1)(S,Q) = cw_1(S,Q)$ respectively. It is easy to verify that G is a $|EC| -$ dimensional linear space with these operations.

For any $S \subseteq N$, let $[S]$ denote the partition of S which consists of the singleton elements of S, i.e., $[S] = \{\{j\} \,|\, j \in S\}$. For $Q \in PT$ and $i \in N$, Q^i denotes the member of Q where i belongs. Additionally, we denote the cardinality of a set by its corresponding lower-case letter, for instance $n = |N|$, $s = |S|$, $q = |Q|$, $q_i = \left|Q^i\right|$ and so on.

A solution is a function $\varphi : G \rightarrow \mathbb{R}^n$ such that $\varphi_i(w)$ is interpreted as the utility payoff which player i should expect from the game w.

Example 1: *As an illustration of the kind of problems that can be modelled with a game with externalities, consider the following game w, which describes the situation where 3 companies ($N = \{1,2,3\}$) are competing for a market. When companies 1, 2, and 3 are on their own, each of them gets a worth of 50 monetary units. However, if any two of them get together, they can take over a larger share of the market and thus make more profit, affecting the other company. Finally, if the grand coalition is formed, they monopolize the entire market and thus obtain the maximum profit of 200 monetary units.*

The intermediate worths are presented in table 1:

Table 1
Set of worths for Example 1

Embedded coalitions	*Worths*
$\{1\},\{2\},\{3\}$	50 50 50
$\{1,2\},\{3\}$	120 40
$\{1,3\},\{2\}$	125 45
$\{2,3\},\{1\}$	130 10
$\{1,2,3\}$	200

Source: Author's elaboration.

Thus, what is the utility payoff which each company should expect from the previous situation?

Now, the group of permutations of N, $S_n = \{\theta : N \to N | \theta \, is \, bijective\}$, acts on EC in the natural way; i.e., for $\theta \in S_n$:

$$\theta\left(S_1, \{S_1, S_2, ..., S_l\}\right) = \left(\theta\left(S_1\right), \{\theta\left(S_1\right), \theta\left(S_2\right), ..., \theta\left(S_l\right)\}\right)$$

And also, S_n acts on the space of payoff vectors, $\mathbb{R}^n$:

$$\theta\left(x_1, x_2, ..., x_n\right) = \left(x_{\theta^{-1}(1)}, x_{\theta^{-1}(2)}, ..., x_{\theta^{-1}(n)}\right)$$

Next, some preliminaries related to integer partitions are needed for subsequent developments.

A partition of a non-negative integer is a way of expressing it as the unordered sum of other positive integers, and it is often written in tuple notation. Formally, $\lambda = [\lambda_1, \lambda_2, ..., \lambda_l]$ is a partition of n if $\lambda_1, \lambda_2, ..., \lambda_l$ are positive integers and $\lambda_1 + \lambda_2 + \cdots + \lambda_l = n$. The set of all partitions of n will be denoted by $\Pi(n)$, and, if $\lambda \in \Pi(n)$, $|\lambda|$ is the number of elements of λ.

For example, the partitions of $n = 4$ are $[1,1,1,1]$, $[2,1,1]$, $[2,2]$, $[3,1]$ and $[4]$. Sometimes we abbreviate this notation by dropping the commas, so $[2,1,1]$ becomes $[211]$.

If $Q \in PT$, there is a unique partition $\lambda_Q \in \Pi(n)$, associated with Q, where the elements of λ_Q are exactly the cardinalities of the elements of Q. In other words, if $Q = \{S_1, S_2, ..., S_m\} \in PT$, then $\lambda_Q = [|S_1|, |S_2|, ..., |S_m|]$.

For a given $\lambda \in \Pi(n)$, we represent by λ° the set of numbers determined by the λ_i's and for $k \in \lambda^\circ$, we denote by m_k^λ the multiplicity of k in partition λ. So, if $\lambda = [4,2,2,1,1,1]$, then $|\lambda| = 6$, $\lambda^\circ = \{1,2,4\}$, and $m_1^\lambda = 3$, $m_2^\lambda = 2$, $m_4^\lambda = 1$.

Let $\lambda, \gamma \in \Pi(n)$ be partitions such that $\gamma^\circ \subseteq \lambda^\circ$, we define the difference $\lambda - \gamma$ as a new partition obtained from λ by removing the elements of γ. For example, $[4,3,2,1,1,1]-[3,1,1]=[4,2,1]$.

2.1 *The axioms*

In the cooperative game theory framework, an axiomatization is an important approach to better understanding cooperative solution concepts. Over the years, many different values and other concepts have been fully characterized by axioms. In particular, we define the usual linearity, symmetry, and efficiency axioms which are asked for solutions to satisfy in the cooperative game theory framework.

Axiom 1 (Linearity) *φ is linear if $\varphi(w_1 + w_2) = \varphi(w_1) + \varphi(w_2)$ and $\varphi(cw_1) = c\varphi(w_1)$, for all $w_1, w_2 \in G$ and $c \in \mathbb{R}$.*

The axiom of linearity means that when a group of players shares the benefits (or costs) stemming from two different issues, how much each player obtains does not depend on whether they consider the two issues together or one by one. Hence, the agenda does not affect the final outcome. Also, the sharing does not depend on the unit used to measure the benefits.

Axiom 2 (Symmetry) *φ is said to be symmetric if and only if $\varphi(\theta \cdot w) = \theta \cdot \varphi(w)$ for every $\theta \in S_n$ and $w \in G$. Here, the game $\theta \cdot w$ is defined as $(\theta \cdot w)(S, Q) = w\left[\theta^{-1}(S, Q)\right]$.*

Symmetry means that players' payoffs do not depend on their names. A player's payoff is only derived from his influence on the worth of the coalitions.

Axiom 3 (Efficiency) *φ is efficient if $\sum_{i \in N} \varphi_i(w) = w(N, \{N\})$ for all $w \in G$.*

We assume that the grand coalition forms, and we leave issues of coalition formation out of this paper. Efficiency means that the value must be feasible and exhaust all the benefits from cooperation, given that everyone cooperates.

A solution that satisfies the previous axioms is referred to as an $LSE solution$.

3. LSE solutions and the transfers procedure

Sánchez-Pérez (2017, Theorem 3) prove that a solution φ satisfies the axioms of linearity, symmetry and efficiency, if there exist $|D_n|$ real numbers $\{\beta(\lambda, j, k) \,|\, (\lambda, j, k) \in D_n\}$ such that,

$$\varphi_i(w) = \frac{w(N,\{N\})}{n} + \sum_{(\lambda,j,k)\in D_n}, \beta(\lambda,j,k)$$

$$\left[\sum_{\substack{(S,Q)\in EC\\ S\ni i,|S|=j\\ \lambda_Q=\lambda}} \sum_{\substack{T\in Q\setminus\{S\}\\ |T|=k}} kw(S,Q) - \sum_{\substack{(S,Q)\in EC\\ S\not\ni i,|S|=j\\ \lambda_Q=\lambda,|Q^i|=k}} jw(S,Q)\right] \quad (1)$$

Here, the set D_n is defined as $D_n = \{(\lambda,j,k)\,|\lambda \in \Pi(n)\setminus\{[n]\}, j \in \lambda^\circ, k \in (\lambda-[j])^\circ\}$.

Although the parameters $\beta = \{\beta(\lambda,j,k)\,|\,(\lambda,j,k) \in D_n\}$ in formula (1) can be any collection of real numbers, if $\beta(\lambda,j,k) \in [0,1]$ we can refer to them as weights (or fractions) of the worths $\{w(S,Q)\}_{(S,Q)\in EC}$.

Now, we describe the final payoff for player $i \in N$ as a result of the following elementary procedure:

1. He/she receives the egalitarian amount $\frac{w(N,\{N\})}{n}$.
2. For each embedded coalition (S,Q), such that $S \neq N$, there are transfers between players in S and players in $N\setminus S$.
   (a) If player i belongs to S, then he/she receives (from each player in each $T \in Q\setminus\{S\}$) a fraction $\beta(\lambda_Q,s,t)$ of the worth $w(S,Q)$. In total from coalition T:

$$t\beta(\lambda_Q,s,t)\cdot w(S,Q)$$

   (b) If player i does not belong to S, then he/she pays (to each player in S) a fraction $\beta(\lambda_Q,s,q_i)$ of the worth $w(S,Q)$. In total to coalition S:

$$s\beta(\lambda_Q,s,q_i)\cdot w(S,Q)$$

Notice that the weights are symmetric in the following sense:

∘ If $i \in S$, then the weights associated to the embedded coalition (S,Q) depend on three parameters: the structure of Q, the cardinality of S and the cardinality of other coalitions T different from S.

∘ If $i \notin S$, the weights depend on: the structure of Q, the cardinality of S and the cardinality of the coalition that contains player i.

In this note, we introduce a new entity that reflects the system of transfers described above. We define for any embedded coalition (S, Q) such that $S \neq N$ and for any fixed $w \in G$, the quantity:

$$A_i^{\beta}(S,Q) = \begin{cases} \sum\limits_{T \in Q \setminus \{S\}} t\beta(\lambda_Q, s, t) \cdot w(S,Q) & if\ i \in S \\ \\ -s\beta(\lambda_Q, s, q_i) \cdot w(S,Q) & if\ i \notin S \end{cases}$$

Thus, if $i \in S$, then $A_i^{\beta}(S,Q) =$ is the amount that player i receives in total from all other coalitions (in Q) different from S. On the other hand, if $i \notin S$, then $A_i^{\beta}(S,Q) =$ is the amount that player i must pay in total to the members in coalition S.

We can now state the main result:

Theorem 1: *Let φ be a value on G, φ is linear, symmetric and efficient if and only if there exists a unique sequence $\{\beta(\lambda, k, j) \mid (\lambda, k, j) \in D_n\}$ such that for any $i \in N$,*

$$\varphi_i^{\beta}(w) = \frac{w(N, \{N\})}{n} + \sum_{\substack{(S,Q) \in EC \\ S \neq N}} A_i^{\beta}(S,Q) \tag{2}$$

Proof. It is immediate; one only needs to substitute the definition of $A_i^{\beta}(S,Q)$ into (2), and re-arrange the terms to obtain (1).

Thus, every linear, symmetric and efficient φ can be obtained from the transfers procedure previously described.

Example 2: *Let $N = \{1, 2, \ldots, 9\}$ be the set of players and take a particular embedded coalition (S, Q), such that $S = \{1, 8\}$ and*

$$Q = \{\{2,6,7\}, \{1,8\}, \{4,9\}, \{3\}, \{5\}\}.$$

Notice that $\lambda_Q = [32211]$ and in order to compute the payoff for player 1 or 8, one has to consider the transfers (associated to the *coalition structure Q) between members of S and members of remaining coalitions $T \in Q \setminus \{S\}$ (table 2).*

Table 2
Transfers among players for a particular embedded coalition

T	*Each member receives from T:*	*Each member of S pays to T:*
$\{2,6,7\}$	$3\beta(\lambda_Q, 2, 3) \cdot w(\{1,8\}, Q)$	$3\beta(\lambda_Q, 3, 2) \cdot w(\{2,6,7\}, Q)$
$\{4,9\}$	$2\beta(\lambda_Q, 2, 2) \cdot w(\{1,8\}, Q)$	$2\beta(\lambda_Q, 2, 2) \cdot w(\{4,9\}, Q)$
$\{3\}$	$\beta(\lambda_Q, 2, 1) \cdot w(\{1,8\}, Q)$	$\beta(\lambda_Q, 1, 2) \cdot w(\{3\}, Q)$
$\{5\}$	$\beta(\lambda_Q, 2, 1) \cdot w(\{1,8\}, Q)$	$(\lambda_Q, 1, 2) \cdot w(\{5\}, Q)$

Source: Author's elaboration.

The next example shows how to obtain any linear, symmetric and efficient solution (for $n = 3$) by applying the transfers procedure.

Example 3: *If $N = \{i, j, k\}$, there are 5 different partitions for the set of players. According to the transfers procedure, the payoff for player i is obtained from the following system of transfers (table 3):*

Table 3
System of transfers for the case n=3

Partition	*Transfers related to player i*
$Q_1 = \{\{i\}, \{j\}, \{k\}\}$	$2\beta([111], 1, 1) \cdot w(\{i\}, Q_1)$ $-\beta([111], 1, 1) \cdot w(\{j\}, Q_1)$ $-\beta([111], 1, 1) \cdot w(\{k\}, Q_1)$
$Q_2 = \{\{i, j\}, \{k\}\}$	$\beta([21], 2, 1) \cdot w(\{i, j\}, Q_2)$ $-\beta([21], 1, 2) \cdot w(\{k\}, Q_2)$
$Q_3 = \{\{i, k\}, \{j\}\}$	$\beta([21], 2, 1) \cdot w(\{i, k\}, Q_2)$ $-\beta([21], 1, 2) \cdot w(\{j\}, Q_2)$
$Q_4 = \{\{j, k\}, \{i\}\}$	$2\beta([21], 1, 2) \cdot w(\{i\}, Q_4)$ $-2\beta([21], 2, 1) \cdot w(\{j, k\}, Q_4)$
$Q_5 = \{\{i, j, k\}\}$	

Source: Author's elaboration.

Thus, re-arranging the terms, any linear, symmetric efficient solution is of the form (for player i):

$$\varphi_i(w) = \frac{w(N,\{N\})}{n}$$

$$+\beta([111],1,1)\left[2w(\{i\},Q_1) - w(\{j\},Q_1) - w(\{k\},Q_1)\right]$$

$$+\beta([21],1,2)\left[2w(\{i\},Q_4) - w(\{j\},Q_3) - w(\{k\},Q_2)\right]$$

$$+\beta([21],2,1)\left[w(\{i,j\},Q_2) + w(\{i,k\},Q_3) - 2w(\{j,k\},Q_4)\right]$$

for any choice of real numbers $\beta([111],1,1)$, $\beta([21],1,2)$, *and* $\beta([21],2,1)$.

In particular, for Example 1, we compute a player's payoff through the transfers procedure.

Example 4: *From the game described in Example 1, we describe the payoff (for player 1, without loss of generality) according to the transfers procedure (table 4).*

Table 4
Game of Example 1

Embedded coalitions	*Worths*
$\{1\},\{2\},\{3\}$	50 50 50
$\{1,2\},\{3\}$	120 40
$\{1,3\},\{2\}$	125 45
$\{2,3\},\{1\}$	130 10
$\{1,2,3\}$	200

Source: Author's elaboration.

- *First, the quantity 200/3 is allocated to player 1.*
- *In partition* $\{\{1\},\{2\},\{3\}\}$, *he/she receives (from players 2 and 3)* $100 \cdot \beta([111],1,1)$ *and pays them* $100 \cdot \beta([111],1,1)$.
- *In partition* $\{\{1,2\},\{3\}\}$, *he/she receives (from player 3)* $120 \cdot \beta([21],2,1)$ *and pays* $40 \cdot \beta([21],1,2)$.

- *In partition* $\{\{1,3\},\{2\}\}$, *he/she receives (from player 2)* $125 \cdot \beta([21],2,1)$ *and pays* $45 \cdot \beta([21],1,2)$.
- *In partition* $\{\{2,3\},\{1\}\}$, *he/she receives (from players 2 and 3)* $20 \cdot \beta([21],1,2)$ *and pays* $260 \cdot \beta([21],2,1)$.

4. Examples of LSE solutions

In this section we briefly present some solutions that can be implemented from the transfers procedure; i.e., they all are of the form (1). As a first example, we take the expected stand-alone value, φ^{ESA}, which tells us how much a player may obtain in a game with externalities when we focus on the stand-alone side of the game:

$$\begin{aligned}&\varphi_i^{ESA}(w) = \tfrac{w(N,\{N\})}{n}\\&+\sum_{\substack{\emptyset\neq S\subset N\\ i\notin S}} \tfrac{s!(n-s-1)!}{n!} w(\{i\},\{S\}\cup N\setminus(S\cup\{i\})]\cup\{\{i\}\})\\&-\sum_{j\in N\setminus\{i\}}\sum_{S\subset N\setminus\{i,j\}} \tfrac{s!(n-s-2)!}{n!} w(\{j\},[N\setminus(S\cup\{i\})]\cup\{S\cup\{i\}\})\end{aligned}$$

which we obtain when we take in (1):

$$\beta(\lambda,s,t) = \begin{cases} \frac{1}{n(n-1)(n-2)} \quad if\ \lambda\in\{m,1,...,1]\}_{m=1}^{n-1}, s=1 \\ and \quad t=m \\ 0 \quad otherwise\end{cases}$$

Shapley value

Pham Do and Norde (2007) define an extension of the Shapley (1953) value to the class of games with externalities as:

$$\varphi_i^{PN}(w) = Sh_i(v),$$

for each $i\in N$ and each $w\in G$, where Sh is the Shapley value operator for transferable utilities (TU) games and v is defined as follows:

$$v(S) = w(S,\{S,[N\setminus S]\})$$

for each $S \subseteq N$.

This solution is of the form (1) with parameters:

$$\beta(\lambda, s, t) = \begin{cases} \frac{(s-1)!(n-s-1)!}{n!} & if\ \lambda \in \{m, 1, ..., 1]\}_{m=1}^{n-1}, s = m \\ and \quad t = 1 & \\ 0 \quad otherwise & \end{cases}$$

Consensus value

Ju (2007) define the consensus value, φ^J, as the middle point between the stand-alone value and the Shapley value of Pham Do and Norde (2007). The corresponding parameters for the consensus value are:

$$\beta(\lambda, s, t) = \begin{cases} \frac{1}{2n(n-2)} & if\ \lambda = [1, 1, ..., 1]\ and\ s = t = 1 \\ \frac{1}{2n(n-1)(n-2)} & if\ \lambda \in \{m, 1, ..., 1]\}_{m=1}^{n-1}, s = 1 \\ and \quad t = m & \\ \frac{(s-1)!(n-s-1)!}{2n!} & if\ \lambda \in \{m, 1, ..., 1]\}_{m=1}^{n-1}, s = m \\ and \quad t = 1 & \\ 0 \quad otherwise & \end{cases}$$

Myerson value

Myerson (1977) proceeds axiomatically and proposes a value extending the well-known Shapley value (Shapley, 1953), which is defined for TU games. His proposal satisfies the axioms of linearity, symmetry, efficiency, and the "null" player property that states that players who have no effect on the outcome should neither receive nor pay anything. The Myerson value of a player is given by:

$$\varphi_i^M(w) = \sum_{(S,Q) \in EC} (-1)^{q-1}(q-1)! \cdot \left(\frac{1}{n} - \sum_{T \in Q \setminus \{S\}, i \notin T} \frac{1}{(q-1)(n-t)} \right) \cdot w(S, Q)$$

which we get with the parameters:

$$\beta(\lambda, s, t) = \frac{(-1)^{|\lambda|}(|\lambda|-1)!}{s}\left(\frac{1}{n} - \sum_{r\in(\lambda-[s,t])^\circ} \frac{m_r^{\lambda-[s,t]}}{(|\lambda|-1)(n-r)}\right)$$

The value of Albizuri et al.

Albizuri *et al.* (2005) obtain a unique value characterized by the properties of linearity, symmetry, efficiency, oligarchy, and an additional symmetry requirement with respect to the embedded coalitions. They define the value for a player as:

$$\varphi_i^{AAR}(w) = \sum_{\substack{(S,Q)\in EC \\ i\in S}} \frac{(s-1)!(n-s)!}{n!P(S,N)} w(S,Q) - \sum_{\substack{(S,Q)\in EC \\ i\in S}} \frac{s!(n-s-1)!}{n!P(S,N)} w(S,Q)$$

where $P(S,N) = |\{(T,Q)\in EC | T = S\}|$. In fact, they notice that $P(S,N) = p(n-s)$, where $p(k)$ represents the number of partitions of any set K with cardinality k.

This solution is also of the form (1). The corresponding parameters are:

$$\beta(\lambda, s, t) = \frac{(n-s-1)!(s-1)!}{n!\cdot p(n-s)}$$

The value of Macho-Stadler et al.

As a final example, Macho-Stadler *et al.* (2007) characterize the value:

$$\varphi_i^{MPW}(w) = \sum_{\substack{(S,Q)\in EC \\ i\in S}} \frac{(s-1)! \prod_{T\in Q\setminus\{S\}} (t-1)!}{n!} w(S,Q)$$

$$- \sum_{\substack{(S,Q)\in EC \\ i\notin S}} \frac{s! \prod_{T\in Q\setminus\{S\}} (t-1)!}{(n-s)\, n!} w(S,Q),$$

which we get when we choose:

$$\beta(\lambda, s, t) = \frac{(s-1)! \prod_{r\in(\lambda-[s])^{\circ}} [(r-1)!]^{m_r^{\lambda-[s]}}}{(n-s)\, n!}.$$

Example 5: *We look at the system of weights for the different solutions described above in the case $n = 3$. For this particular case, there are 3 different weights associated with the transfers procedure:*

Table 5
System of weights for LSE solutions for $n = 3$

Weights	φ^{ESA}	φ^{PN}	φ^{J}	φ^{M}	φ^{AAR}	φ^{MPW}
$\beta([111], 1, 1)$	1/6	1/6	1/6	-1/6	1/12	1/12
$\beta([21], 1, 2)$	1/6	0	1/12	1/3	1/12	1/12
$\beta([21], 2, 1)$	0	1/6	1/12	1/6	1/6	1/6

Source: Author's elaboration.

Now, for the case $n = 4$ there are 7 different weights associated with the transfers procedure (table 6):

Table 6
System of weights for LSE solutions for $n = 4$

Weights	φ^{ESA}	φ^{PN}	φ^{J}	φ^{M}	φ^{AAR}	φ^{MPW}
$\beta([1111], 1, 1)$	1/24	1/12	1/16	1/6	1/60	1/72
$\beta([211], 2, 1)$	0	1/24	1/48	-1/12	1/48	1/48
$\beta([211], 1, 2)$	1/24	0	1/48	-1/6	1/60	1/72
$\beta([211], 1, 1)$	0	0	0	0	1/60	1/72
$\beta([31], 3, 1)$	0	1/12	1/24	1/12	1/12	1/12
$\beta([31], 1, 3)$	1/24	0	1/48	1/4	1/60	1/36
$\beta([22], 2, 2)$	0	0	0	1/8	1/48	1/48

Source: Author's elaboration.

As we can observe, in both cases, the Myerson value is the only solution that considers negative weights.

Example 6: *Recall the game described in Example 1, and applying the previous solutions, we get the following payoffs.*

Table 7
Computation of LSE solutions for Example 1

	Solution					
Player	φ^{ESA}	φ^{PN}	φ^{J}	φ^{M}	φ^{AAR}	φ^{MPW}
1	55.83	64.17	60	42.5	58.75	58.75
2	73.33	66.67	70	80	70	70
3	70.83	69.17	70	77.5	71.25	71.25

Source: Author's elaboration.

In order to select one of these solutions, one have to consider the axioms (other than linearity, symmetry and efficiency) that characterize each value. Some axioms are widely accepted and others could be more controversial.

For instance with the Myerson value, player 1 obtains a payoff of 42.5, which is less than his/her individual worth in the game (assuming all players are on their own). This might be a consequence of *the "null" property considered in the characterization of the Myerson value.*

5. Final comments

The family of solutions for games with externalities that are simultaneously linear, symmetric, and efficient is wide and describes a $|D_n|-$ dimensional affine vector space. It contains, among others, the expected stand-alone value, the Myerson value (Myerson, 1977), the Shapley value (Pham Do and Norde, 2007), the consensus value (Ju, 2007), the value of Albizuri *et al.* (2005), and the value of Macho-Stadler *et al.* (2007). A better understanding pattern of solutions is essential to get the difference (and similarity) between solutions within a given class and to know what changes when we switch to a different class of solutions. The paper unveils how any such class reflects a policy for sharing the common surplus in terms of transfers between players of a fixed embedded coalition and players of every coalition that belongs to the same partition.

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