A NOTE ABOUT THE DIFFERENT CHARACTERIZATIONS OF THE EXPECTED UTILITY THEOREM

UNA NOTA SOBRE LAS DIFERENTES CARACTERIZACIONES DEL TEOREMA DE LA UTILIDAD ESPERADA

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Resumen: En este trabajo se muestra la equivalencia axiomática entre las distintas caracterizaciones del Teorema de la Utilidad Esperada propuestas por Mas-Colell *et al.* (1995), Jehle y Reny (2011), Maschler *et al.* (2013) y Rubinstein (2012). Se utiliza un lenguaje general que unifica la notación y se propone una definición recursiva del espacio de loterías de la Teoría de la Utilidad Esperada.

Abstract: In this work, we prove the axiomatic equivalence of several characterizations of the Expected Utility Theorem proposed by Mas-Colell *et al.* (1995), Jehle and Reny (2011), Maschler *et al.* (2013), and Rubinstein (2012). A general language is used for unifying the notation, and we introduce a recursive definition of the lottery space in the Expected Utility Theory.

Clasificación JEL/JEL Classification: D810, D000, C700

Palabras clave/keywords: expected utility theory, lottery space, axiomatic equivalence, teoría de la utilidad esperada, espacio de loterías, equivalencia axiomática

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1. Introduction

John von Neumann and Oskar Morgenstern developed the Expected Utility Theory (EUT) in 1944. This theory has been extensively used in game theory, industrial organization, financial theory, and other important areas in economic theory. Graduate courses use texts like Mas-Colell *et al.* (1995), Jehle and Reny (2011), Maschler *et al.* (2013), Rubinstein (2012), among others, to explain it. The axiomatic approach on each text apparently uses different languages and axioms. In this paper, we prove the equivalence among those axiomatic systems, and build the lottery space in a recursive way. The importance of doing this homologation consists on the unification in a common language of properties and systems that look different from a first sight.

Although the equivalence between axiomatizations is almost obvious for specialists, it is not so for most students. That is why we believe in the need to unify seemingly different languages and to offer a formal proof of the equivalence between several axiomatizations. In general, even slight modifications of axioms can generate different theories. A classic example is non-Euclidean geometries.

The expected utility theory has been heavily revised. The well-known Allais paradox showed the inconsistency of the theory with some cases of agents' choices faced with specific lotteries. This has made it possible to analyse the consequences of weaknesses and variants of the independence axiom. Several authors have proposed alternative schemes to model decisions under risk and uncertainty. Its study is beyond the scope of this note. We recommend consulting Barbera *et al.* (2004), and specially Sudgen (2004) and Schmidt (2004).

The standard proof of the expected utility theorem relies heavily on the axiom of continuity to construct the utility of a lottery. Given an arbitrary initial lottery, we construct its utility considering a special lottery that is indifferent to the initial one. The special lottery is a simple lottery that combines the most preferred with the least preferred result. The utility of the initial lottery is the weight that has the most desired result. The axiom of independence is the key to obtain that the numerical representation is a linear mapping on the probabilities; that is, it has the fundamental property of the expected utility.

Cantala (2007) describes an interesting characterization of Expected Utility. The author translates preferences over lotteries into preferences over "shifts in probabilities". Changes in probabilities are relatively straightforward. The probability of one outcome is reduced by a fixed feasible amount and the same amount increases the probability of another outcome. The author constructs an independence axiom suitable to represent these types of "jumps" in lotteries. He obtains a new characterization of the expected utility: continuous preferences over lotteries are represented by an expected utility function if and only if the induced preferences satisfy his version of the independence axiom for preferences over shifts in probabilities. The axiom means that the ordering of shifts in probability is independent of the weight and the original lottery. Cantala's treatment (2007) can be repeated in any of the axiomatic versions. It is possible to use his proof technique to generate new versions of the expected utility theorem by adapting the new version of the independence axiom on each axiomatic system. In this paper, we are interested in the equivalences among the different axiomatic systems rather than in the constructive proof of the expected utility theorem. The approach of Cantala (2007) is relevant because "it makes it clear that the cardinal nature of expect utility is only a behavioural characteristic rooted in the independence axiom and has no normative appeal" (Cantala, 2007: 101).

We show a proof based strongly on the construction of a common language for rewriting the different systems of axioms. This means the specification of a common lottery space where the preferences are defined. The recursive construction of the lottery space is necessary: it is not an exaggeration. It is introduced for technical reasons of consistency, it makes sense, and it is suggested in Jehle and Reny (2011).

Compound lotteries can be classified by their level of complexity. Every compound lottery of any level has an expected utility level to compare with any other lottery. However, this comparison requires introducing an axiom that makes it possible to relate a compound lottery with a simple one of the first level. This is done in the Jehle and Reny (2011) axiomatization and Maschler *et al.* (2013). These two axiomatizations are the better ones for explaining the economic content of the axioms, in addition to the axioms of rationality of preferences. The axiomatic presentations of Mas-Colell *et al.* (1995) and Rubinstein (2012) focus on the two essential axioms in the proof of the expected utility theorem: independence and continuity. In their works, they consider compound lotteries, but they are artificially reduced to those of the first level. They avoid an axiom of simplification by introducing the convexity of the lottery space.

The simplification axiom from compound to simple lotteries is an empirically unsustainable assumption. It is not obvious that an agent recognizes a fairly complex compound lottery as indifferent to its simplified form. In Mas-Colell *et al.* (1995), a simplex is introduced as the lottery space. The convexity of this space makes it closed under convex combinations. But this is a mathematical reason that artificially simplifies and reduces (by convexity) the meaning of a compound lottery of a second or more level. The simplification axiom is strong but important because it considers the levels of complexity in the construction of lotteries.

In summary, the presentation by Mas-Colell *et al.* (1995) is more classical, but its continuity axiom is apparently more complex. We could say that it abuses mathematics somewhat and sacrifices part of the economic meaning. On the other hand, the presentations by Jehle and Reny (1911) and Maschler *et al.* (2013) are more careful. Their economic content has greater detail. The formulation of the continuity axiom is clearer. The first one even breaks down independence into two apparently less strong axioms: substitution and reduction to simple games.

The order of the paper is as follows: in section 2, we analyze the similarities in notation and concepts of the four axiomatic systems, establishing a common language regarding the concept of compound lotteries and the simplification action. Then, we compare two pairs of axiomatic systems due to the similarities between them: Maschler *et al.* (2013) and Jehle and Reny (2011) with four axioms each, and Mas-Colell *et al.* (1995) and Rubinstein (2012), each one with a system of two axioms. Finally, we show the equivalence of the four systems by means of a theorem in section 3 of this article.

2. Model

An agent must take a decision before a random event occurs. The final consequences of his decision are represented as possible outcomes in the set $X = \{x_1, \ldots, x_K\}$. The objects that each agent values are called *lotteries* and are listed such that $L = [p_1(x_1), \ldots, \; p_K(x_K)]$ is a *simple lottery* with $p_k \geq 0$ and $\sum_{k=1}^{K} p_k = 1$. L denotes the lottery that assigns probability p_k to each consequence $x_k \in X$. For simplicity, we will denote the lottery $[\alpha(x_1), 0(x_2), \ldots, 0(x_{K-1}), (1-\alpha)(x_K)]$ just by $[\alpha(x_1), (1-\alpha)(x_K)]$ for $\alpha \in (0, 1)$.

Let $\mathcal{L}_1 = \{ [p_1(x_1), \ldots, \; p_K(x_K)] \, | x_1, \ldots, x_K \in X; \; p_k \geq 0 \; and \; \sum_{k=1}^{K} p_k = 1 \}$ be the set of simple lotteries, and $\mathcal{L}_{n+1} = \mathcal{L}_n U$

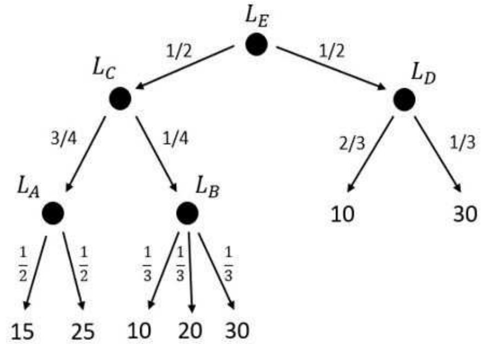
$\{ \ [q_1 (L_1),\ q_2 (L_2), \ldots,\ q_H (L_H)] \mid \ L_h \in \mathcal{L}_n;\ q_h \geq 0$ *and* $\sum_{h=1}^{H} q_h = 1,\ \forall h = 1,\ 2,\ \ldots, H \}$. Then, the *space of lotteries* is given by $\mathcal{L} = \overset{\infty}{\underset{n=1}{U}} \mathcal{L}_n$. The lotteries that have the particularity of granting a consequence $x_k \in X$ with probability equal to one will be called *degenerated lotteries* and we denote them by $L_{x_k} = [1\,(x_k)]$. Lotteries that have the characteristic of having other lotteries as consequences will be called *compound lotteries*, that is, elements in $\mathcal{L}_n$ with $n \geq 2$.

Example: In order to illustrate the previous ideas, let us consider the next lotteries:

L_A = [(1/2)(15), (1/2)(25)]
L_B = [(1/3)(10), (1/3)(20),(1/3)(30)]
L_C = [(3/4)(LA), (1/4)(LB)]
L_D = [(2/3)(10), (1/3)(30)]
L_E = [(1/2)(LC), (1/2)(LD)]
L = [(6/16)(10), (3/16)(15), (1/24)(20), (3/16)(25), (5/24)(30)]

Notice that $L, L_A, L_B, L_D \in \mathcal{L}_1, L_C \in \mathcal{L}_2, L_E \in \mathcal{L}_3$. We can represent the process of composing the lotteries with the tree scheme given in figure 1.

Figure 1
Scheme of the composing of lotteries over a set of outcomes (the terminal nodes)



Source: Authors' elaboration.

A classical axiom about the simplification of lotteries says that each compound lottery must be indifferent to a simple lottery. In this case, lottery L is indifferent to lottery L_E, because the probability of each outcome is the same in both lotteries. A strong axiom about individual preferences is avoided if we assume convexity of the lottery space. However, its economic meaning is important.

For comparing the four axiomatic systems, we define a general binary relation $\succsim$ on $\mathcal{L}$, which allows us to compare lotteries. Such relation is called a *preference relation*.

A preference relation $\succsim$ is rational if:

1. For every $L, L' \in \mathcal{L} \times \mathcal{L}$, we have $L \succsim L'$ or $L' \succsim L$.
2. For any $L, L', L'' \in \mathcal{L}$, if $L \succsim L'$ and $L' \succsim L''$, then $L \succsim L''$, where $L \succsim L'$ means that L is at least as good as L'.[1]

According to the definition of compound lottery, it is possible to have uncertain decision problems where the consequences of the alternatives are uncertain objects. For example, we have a raffle such that one of the possible prizes is a chance to participate in another raffle. So, this idea motivates the concept of simplification of compound lotteries into an equivalent simple lottery, in the sense of the preference relation. As the notion of simplification belongs to the four axiomatic systems under our study, it is convenient to have a common notation for it.

We define the lottery simplification, $Simp : \mathcal{L} \to \mathcal{L}_1$, for $n \geq 2$, as follows:

$$Simp\,(L) = \begin{matrix} L \;\; if \; L \in \mathcal{L}_1 \\ \left[r_1\,(x_1)\,, \ldots, r_K\,(x_K)\right] \;\; if \; L \in \mathcal{L}_n \end{matrix} \tag{1}$$

where $L = [q_1\,(L_1)\,, \ldots, q_M\,(L_M)]$ with $L_1, \ldots, L_M \in \overset{n-1}{\underset{i=1}{U}} \mathcal{L}_i$; we assume that $Simp\,(L_1) = \left[p_1^1\,(x_1)\,, \ldots, p_K^1\,(x_K)\right]$, $Simp\,(L_2) = \left[p_1^2\,(x_1)\,, \ldots, p_K^2\,(x_K)\right]$, $Simp\,(L_M) = \left[p_1^M\,(x_1)\,, \ldots, p_K^M\,(x_K)\right]$ and we have, for every $k \in \{1, \ldots, K\}$, $r_k = q_1 p_k^1 + q_2 p_k^2 + \ldots + q_M p_k^M$.

[1] The *strong preference* and *indifference* are defined in the usual way, as follows: $L \succ L' \;\Leftrightarrow\; L \succsim L'$ but not $L' \succsim L$; $L \approx L' \;\Leftrightarrow\; L \succsim L'$ and $L' \succsim L$.

As we said before, each axiomatic system under our study assumes that a compound lottery can be simplified. In Jehle and Reny (2011) and Maschler *et al.* (2013), this fact is assured by axioms *G6* and *S*, respectively. In Mas-Colell *et al.* (1995) and Rubinstein (2012), the definition of compound lottery allows to make this kind of simplification.

2.1 *Axiomatic systems*

Now, we introduce the four axiomatic systems of our study using the notation of the previous section. We are going to refer to each one of them by its author.

Mas-Colell et al. (1995):

- *Independence* (I_{MC}): For every $L, L', L'' \in \mathcal{L}$ and $\alpha \in (0,1)$, $L \succsim L'$ if and only if $[\alpha(L), (1-\alpha)(L'')] \succsim [\alpha(L'), (1-\alpha)(L'')]$.
- *Continuity* (C_{MC}): For every $L, L', L'' \in \mathcal{L}$, the following sets are closed.
  $\{\alpha \in [0,1] \mid [\alpha(L), (1-\alpha)(L'')] \succsim L'\}$
  $\{\alpha \in [0,1] \mid L' \succsim [\alpha(L), (1-\alpha)(L'')]\}$

Maschler et al. (2013):

- *Simplification* (S): For every $L \in \mathcal{L}$, $L \approx Simp(L)$.
- *Independence* (I): If $L = [q_1(L_1), \ldots, q_m(L_m), \ldots, q_M(L_M)] \in \mathcal{L}$ and $L^* \approx L_m$, then $L \approx [q_1(L_1), \ldots, q_m(L^*), \ldots, q_M(L_M)]$.
- *Continuity* (C): For every $L, L', L'' \in \mathcal{L}$ such that $L \succsim L' \succsim L''$, there exists $\alpha \in [0,1]$, where $L' \approx [\alpha(L), (1-\alpha)(L'')]$.
- *Monotonicity* (M): Assuming $\alpha, \beta \in [0,1]$ and $L \succ L'$, then, $[\alpha(L), (1-\alpha)(L')] \succsim [\beta(L), (1-\beta)(L')]$ if and only if $\alpha \geq \beta$.

Jehle and Reny (2011):

- *Continuity* ($G3$): For every $L \in \mathcal{L}$, there exists a real number $\alpha \in [0,1]$, such that $L \approx [\alpha(L_{x_K}), (1-\alpha)(L_{x_1})]$.
- *Monotonicity* ($G4$): For every $\alpha, \beta \in [0,1]$, $[\alpha(L_{x_K}), (1-\alpha)(L_{x_1})] \succsim [\beta(L_{x_K}), (1-\beta)(L_{x_1})]$ if and only if $\alpha \geq \beta$.
- *Substitution* ($G5$): If $L = [p_1(L_1), \ldots, p_M(L_M)]$ and $L' = [p_1(L'_1), \ldots, p_M(L'_M)]$ with $L_m \approx L'_M$ for every m, then $L \approx L'$.
- *Reduction to simple gambles* ($G6$): For every $L \in \mathcal{L}$, if $Simp(L) = [r_1(x_1), \ldots, r_K(x_K)]$, then $L \approx Simp(L)$, where x_1 denotes the worst possible consequence and x_k denotes the best one.

Rubinstein (2012):

- *Independency* (I_R): For any $L, L', L'' \in \mathcal{L}$ and $\alpha \in (0,1)$, $L \succsim L'$ if and only if $[\alpha(L), (1-\alpha)(L'')] \succsim [\alpha(L'), (1-\alpha)(L'')]$.
- *Continuity* (C_R): If $L \succ L' \succ L''$, there exists a real number $\alpha \in (0,1)$ such that $L' \approx [\alpha(L), (1-\alpha)(L'')]$.

Independence axiom by Maschler *et al.* (2013), and Jehle and Reny's (2011) *Substitution* one imply that two lotteries are equivalent if they assign the same probability distribution to different consequences but indifferent between them. That is, if you create a new lottery taking an existing one but substituting a consequence with an indifferent one, both lotteries are indifferent, too.

In all axiomatic systems, the *Continuity* axiom allows the representation of any lottery using a better and a worse lottery.

The *Monotonicity* axiom, in Jehle and Reny (2011) and Maschler *et al.* (2013) axiomatic systems, implies to prefer lotteries that assign a high probability to more preferred consequences.

3. Axiomatic equivalencies

First, we analyze the *Simplification* axiom; notice that this property is part of the Maschler *et al.* (2013), and Jehle and Reny (2011) axiomatic systems (S and $G6$, respectively), and the equivalence between them is straightforward. In the Mas-Colell *et al.* (1995) and Rubinstein (2012) axiomatic systems, this property is implicit in the definition of a compound lottery. In Mas-Colell *et al.* (1995), a compound lottery is denoted by $(L_1, \ldots, L_M; \alpha_1, \ldots, \alpha_M)$ and its simplification is given by $(q_1, \ldots, q_K)$ where, for every $k = 1, \ldots, K$, $q_k = \alpha_1 p_k^1 + \alpha_2 p_k^2 + \ldots + \alpha_M p_k^M$ denotes the probability under $(L_1, \ldots, L_M; \alpha_1, \ldots, \alpha_M)$ of the consequence x_k. That construction implies the generalization for any level of composition as long as the probability distribution of the compound lottery is well defined. The construction of compound lotteries in Rubinstein (2011) is given in a similar way.

On the other hand, the axiomatic system of Jehle and Reny (2011) has six axioms. Two of them are included in our definition of preference relation: the completeness axiom $G1$ and the transitivity axiom $G2$. Also, the simplification axiom $G6$ was rewritten according to our notation and the function *Simp*. The equivalence between S and $G6$ is straightforward. Henceforth, we conclude that the simplification axiom is present in the four axiomatic systems.

Now, we show the main result of our work.

Theorem: The following statements are equivalent.

1. The preference relation $\succsim$ on $\mathcal{L}$ satisfies the I_{MC} and C_{MC} axioms.
2. The preference relation $\succsim$ on $\mathcal{L}$ satisfies the I, C and M axioms.
3. The preference relation $\succsim$ on $\mathcal{L}$ satisfies the $G3$, $G4$, and $G5$ axioms.
4. The preference relation $\succsim$ on $\mathcal{L}$ satisfies the I_R and C_R axioms.

We use the following lemma.[2]

Lemma:
If $\succsim$ on $\mathcal{L}$ satisfies the I_{MC} axiom, then for every $\alpha \in (0,1)$ and $L,\ L', L'' \in \mathcal{L}$, $L \succ L'$ if and only if $[\alpha\,(L)\,,\ (1-\alpha)\,(L'')\,]\succ[\,\alpha\,(L')\,,\ (1-\alpha)\,(L'')]$, and $L \approx L'$ if and only if $[\alpha\,(L)\,,\ (1-\alpha)\,(L'')\,]\approx[\,\alpha\,(L')\,,\ (1-\alpha)\,(L'')]$.

Proof of the theorem:

First, we are going to show that Statement 1 implies Statement 2; that is, if $\succsim$ satisfies I_{MC} and C_{MC}, then satisfies I, C and M.

For proving that $\succsim$ satisfies I, we must show that:

$$if L = [q_1\,(L_1)\,, \ldots, q_m\,(L_m)\,, \ldots, q_M\,(L_M)] \in \mathcal{L}$$

and $L_m \approx L^*$, then $L \approx [q_1\,(L_1)\,, \ldots, q_m\,(L^*)\,, \ldots, q_M\,(L_M)]$.

We define:

$$L^I = [\,\frac{q_1}{\sum_{i\neq m} q_i}\,(L_1), \ldots, \frac{q_{m-1}}{\sum_{i\neq m} q_i}\,(L_{m-1})\,, \frac{q_{m+1}}{\sum_{i\neq m} q_i}\,(L_{m+1})\,, \ldots, \frac{q_M}{\sum_{i\neq m} q_i}\,(L_M)] \tag{2}$$

Because of the Lemma, we have $\left[q_m\,(L_m)\,, (1-q_m)\left(L^I\right)\right] \approx \left[q_m\,(L^*)\,, (1-q_m)\left(L^I\right)\right]$.

[2] Hara *et al.* (1997). Solution manual of Mas-Colell *et al.* (1995).

Notice that the lottery at the left of the previous relation is equal to L, and the simplification of the lottery at the right is:

$$[q_1(L_1), \ldots, q_m(L^*), \ldots, q_M(L_M)]$$

Then, because of S and the transitivity of $\approx$ we have:

$$L \approx [q_1(L_1), \ldots, q_m(L^*), \ldots, q_M(L_M)]$$

For proving that $\succsim$ satisfies C, we must show that for any three lotteries $L, L', L'' \in \mathcal{L}$ such that $L \succsim L' \succsim L''$, there exists a number $\alpha \in [0,1]$, where $L' \approx [\alpha(L),\ (1-\alpha)(L'')]$.

Let $L, L', L'' \in \mathcal{L}$ be lotteries such that $L \succsim L' \succsim L''$. We define:

$$A^+ = \{\alpha \in [0,1] \,|\, [\alpha(L),\ (1-\alpha)(L'')] \succsim L'\} \text{ and}$$

$$A^- = \{\alpha \in [0,1] \,| L' \succsim [\alpha(L),\ (1-\alpha)(L'')]\}.$$

Because of C_{MC}, A^+ and A^- are closed sets on $[0,1]$, and then $A^+,\ A^- \subset [0,1]$. Due to the completeness of $\succsim$, $[0,1] \subset A^+ \cup A^-$ and then, $[0,1] = A^+ \cup A^-$. If $A^+ \cap A^- = \emptyset$, there is a contradiction about the connection of $[0,1]$. So $A^+ \cap A^- \neq \emptyset$, which means that there exists α such that $L' \approx [\alpha(L),\ (1-\alpha)(L'')]$.

To show that if $\succsim$ satisfies C_{MC} and I_{MC}, then $\succsim$ satisfies M, we must prove that for any $\alpha, \beta \in [0,1]$ and $L,\ L' \in \mathcal{L}$ such that $L \succ L'$ we must have $[\alpha(L), (1-\alpha)(L')] \succsim [\beta(L), (1-\beta)(L')]$ if and only if $\alpha \geq \beta$, but this result is already done.[3]

$\diamond$

Now, we will show that Statement 2 implies Statement 3; that is, if $\succsim$ on $\mathcal{L}$ satisfies I, C and M, then it satisfies $G3$, $G4$, and $G5$. Notice that $G3$ is implied directly from C, and $G4$ is implied by M. Moreover, $G5$ is implied by repeated applications of I plus the transitivity of $\succsim$; to see this fact, consider $L, L' \in \mathcal{L}$ such that $L = [p_1(L_1), \ldots, p_M(L_M)]$, $L' = [p_1(L'_1), \ldots, p_M(L'_M)]$, and $L_m \approx L'_M$ for every $m = 1, \ldots, M$. Applying I M times, we have:

$$[p_1(L_1), \ldots, p_M(L_M)] \approx [p_1(L'_1), \ldots, p_M(L_M)] \approx$$
$$[p_1(L'_1), p_2(L'_2), \ldots, p_M(L_M)] \approx \ldots \approx [p_1(L'_1), \ldots, p_M(L'_M)],$$

[3] This result is proven in Mas-Colell *et al.* (1995: 176).

and, because of the transitivity of the relation:

$$[p_1(L_1), \ldots, p_M(L_M)] \approx [p_1(L'_1), \ldots, p_M(L'_M)].$$

$\diamond$

Now, we are going to show that Statement 3 implies Statement 4. That is, if $\succsim$ on $\mathcal{L}$ satisfies $G3$, $G4$, and $G5$, then it satisfies $C_R y I_R$.

For showing that $\succsim$ on $\mathcal{L}$ satisfies C_R (that is, for every $L,\ L',\ L'' \in \mathcal{L}$ such that $L \succ L' \succ L''$, there exists $\alpha \in (0,1)$, such that $L' \approx [\alpha(L), (1-\alpha)(L'')]$), let $L, L', L'' \in \mathcal{L}$ be three lotteries such that $L \succ L^{'} \succ L''$. Because of $G3$, there exist $\beta,\ \gamma,\ \delta \in (0,1)$, such that $L \approx [\beta(L_{x_K}), (1-\beta)(L_{x_1})]$, $L' \approx [\gamma(L_{x_K}), (1-\gamma)(L_{x_1})]$, and $L'' \approx [\delta(L_{x_K}), (1-\delta)(L_{x_1})]$. Notice that, because of $G4$ and $L \succ L' \succ L''$, we have $\beta > \gamma > \delta$; so, we can construct a lottery $L^* \in \mathcal{L}$ such that $L^* = [\alpha\ [\beta(L_{x_K}), (1-\beta)(L_{x_1})]), (1-\alpha)([\delta(L_{x_K}), (1-\delta)(L_{x_1})])]$, with $\alpha = \frac{\gamma-\delta}{\beta-\delta}$, and then, $Simp(L^*) = [\gamma(L_{x_K}), (1-\gamma)(L_{x_1})]$. Thus, because of $G5$, we have $L^* \approx [\alpha(L), (1-\alpha)(L'')]$, and by transitivity and $Simp(L^*) \approx L'$, it is implied that $L' \approx [\alpha(L), (1-\alpha)(L'')]$, and this is the fact we want to prove.

For showing that $\succsim$ on $\mathcal{L}$ satisfies I_R, we must prove that for every $L, L', L'' \in \mathcal{L}$ and $\alpha \in (0,1)$ we have $L \succsim L'$ if and only if $[\alpha(L),\ (1-\alpha)(L'')] \succsim [\alpha(L'),\ (1-\alpha)(L'')]$. First, let us assume $L \succsim L'$, and we must prove that $\alpha(L),\ (1-\alpha)(L'')] \succsim [\alpha(L'),\ (1-\alpha)(L'')]$.

If L'' is an arbitrary lottery, there are three cases to study: a) $L \succsim L'' \succsim L'$, b) $L'' \succsim L \succsim L'$, and c) $L \succsim L' \succsim L''$.

Case a): We assume $L \succsim L'' \succsim L'$

If $L \approx L'$, then $L \approx L' \approx L''$. Because of $G5$, we have:

$$[\alpha(L),\ (1-\alpha)(L'')] \approx [\alpha(L'),\ (1-\alpha)(L'')].$$

By definition of $\approx$ we have:

$$[\alpha(L),\ (1-\alpha)(L'')] \succsim [\alpha(L'),\ (1-\alpha)(L'')].$$

Now, if $L \approx L'$ is not true, then $L \succ L'$. Because of $G3$, there exist real numbers $\beta,\ \gamma,\ \delta \in [0,1]$ such that $L \approx [\beta(L_{x_K}), (1-\beta)(L_{x_1})]$, $L'' \approx [\gamma(L_{x_K}), (1-\gamma)(L_{x_1})]$, and $L' \approx [\delta(L_{x_K}), (1-\delta)(L_{x_1})]$. Because of $G4$ and $L \succsim L^{''} \succsim L'$, we have $\beta \geq \gamma \geq \delta$.

We need to compare $[\alpha(L), (1-\alpha) L'']$ with $[\alpha(L'), (1-\alpha) L'']$, where $\alpha \in [0,1]$. Applying $G5$ and $G6$ to these expressions, we have:

$$\begin{gathered}[\alpha(L), (1-\alpha) L''] \\ \approx [\alpha([\beta(L_{x_K}), (1-\beta)(L_{x_1})]),\ (1-\alpha)[\gamma(L_{x_K}), (1-\gamma)(L_{x_1})]] \\ \approx [(\alpha\beta + (1-\alpha)\gamma)(L_{x_K}),\ \alpha(1-\beta) + (1-\alpha)(1-\gamma)(L_{x_1})]\end{gathered} \tag{3}$$

$$\begin{gathered}[\alpha(L'), (1-\alpha) L''] \\ \approx [\alpha([\delta(L_{x_K}), (1-\delta)(L_{x_1})]),\ (1-\alpha)[\gamma(L_{x_K}), (1-\gamma)(L_{x_1})]] \\ \approx [(\alpha\delta + (1-\alpha)\gamma)(L_{x_K}),\ \alpha(1-\delta) + (1-\alpha)(1-\gamma)(L_{x_1})]\end{gathered} \tag{4}$$

Because of $G4$ and the transitivity of $\succsim$, we have $[\alpha(L), (1-\alpha) L''] \succsim [\alpha(L'), (1-\alpha) L'']$, so the proof is done.

The proofs for b) and c) are analogous to a).

Now, we have to prove that:

$$[\alpha(L),\ (1-\alpha)(L'')] \succsim [\alpha(L'),\ (1-\alpha)(L'')]$$

implies $L \succsim L'$. By contradiction $L \succsim L'$, we have $L' \succ L$. Thus, we have to check three cases: a) $L'' \succsim L' \succ L$, b) $L' \succ L \succsim L''$, and c) $L' \succsim L'' \succ L$.

Case a) $L'' \succsim L' \succ L$

By hypothesis of case a) and the transitivity of $\succ$, we have $L'' \succ L$. Because of $G3$, there exist real numbers $\beta,\ \gamma,\ \delta \in [0,1]$ such that $L'' \approx [\beta(L_{x_K}), (1-\beta)(L_{x_1})]$, $L' \approx [\gamma(L_{x_K}), (1-\gamma)(L_{x_1})]$, and $L \approx [\delta(L_{x_K}), (1-\delta)(L_{x_1})]$. Because of $G4$ and $L'' \succsim L' \succ L$, we have $\beta \geq \gamma > \delta$. If we compare $[\alpha(L), (1-\alpha) L'']$ with $[\alpha(L'), (1-\alpha) L'']$, where $\alpha \in [0,1]$, and applying $G5$ and $G6$ we have:

$$\begin{gathered}[\alpha(L), (1-\alpha) L''] \\ \approx [\alpha([\delta(L_{x_K}), (1-\delta)(L_{x_1})]),\ (1-\alpha)[\beta(L_{x_K}), (1-\beta)(L_{x_1})]] \\ \approx [(\alpha\delta + (1-\alpha)\beta)(L_{x_K}),\ \alpha(1-\delta) + (1-\alpha)(1-\beta)(L_{x_1})]\end{gathered}$$

$$(5)$$

$$\begin{gathered}[\alpha\left(L'\right),(1-\alpha)\,L''] \\ \approx[\alpha\left([\gamma\left(L_{x_K}\right),(1-\gamma)\left(L_{x_1}\right)]\right),\ (1-\alpha)\left[\beta\left(L_{x_K}\right),(1-\beta)\left(L_{x_1}\right)\right]] \\ \approx[(\alpha\gamma+(1-\alpha)\,\beta)\left(L_{x_K}\right),\ \alpha\,(1-\gamma)+(1-\alpha)\,(1-\beta)\left(L_{x_1}\right)]\end{gathered} \tag{6}$$

This expression means a contradiction to the hypothesis. So, the correct one must be $L \succsim L'$. The proof of cases b) and c) are analogous.

$\diamond$

Now we are going to show that Statement 4 implies Statement 1; that is, $\succsim$ on $\mathcal{L}$ satisfies C_R and I_R, then it satisfies C_{MC} and I_{MC}. First, notice that I_{MC} and I_R are completely equivalents. Now we show that if $\succsim$ on $\mathcal{L}$ satisfies I_R and C_R, then it satisfies C_{MC}. For proving that $\succsim$ satisfies C_{MC}, we must prove that for every $L, L', L'' \in \mathcal{L}$, the sets:

$$\{\alpha\in[0,1]\,|\,[\alpha\left(L\right),(1-\alpha)\left(L''\right)]\succsim L'\}$$

and

$$\left\{\alpha\in[0,1]\,|L'\succsim\left[\alpha\left(L\right),(1-\alpha)\left(L^{''}\right)\right]\right\}$$

are closed ones in $[0,1]$. Let $L,\ L',\ L''\in\mathcal{L}$ be arbitrary lotteries. Define:

$$A^{+}=\{\alpha\in[0,1]\,|\,[\alpha\left(L\right),\ (1-\alpha)\left(L''\right)]\succsim L'\}$$

and

$$A^{-}=\{\alpha\in[0,1]\,|L'\succsim[\alpha\left(L\right),\ (1-\alpha)\left(L''\right)]\}\,. \tag{7}$$

Let us analyze three cases:

a) $L\succsim L'$ *and* $L''\succsim L'$

If $L \succ L'$ and $L'' \succ L'$, then $A^{-}=\emptyset$ is a closed set as well as $A^{+}=[0,1]$. If $L\approx L'$ and $L''\succ L'$, then $A^{-}=\{1\}$ and $A^{+}=[0,1]$

are closed sets. If $L \succ L'$ and $L'' \approx L'$, then $A^- = \{0\}$ and $A^+ = [0, 1]$ are closed sets, too. If $L \approx L'$ and $L'' \approx L'$, then $A^- = [0, 1]$ and $A^+ = [0, 1]$ are closed sets, again.

b) $L' \succsim L$ *and* $L' \succsim L''$

If $L' \succ L$ and $L' \succ L''$, then $A^- = [0, 1]$ and $A^+ = \emptyset$. If $L' \approx L$ and $L' \succ L''$, then $A^- = [0, 1]$ and $A^+ = \{1\}$. If $L' \succ L$ and $L' \approx L''$, then $A^- = [0, 1]$ and $A^+ = \{0\}$. In all these cases, both sets are closed.

c) $L \succ L' \succ L''$

Because of C_R, there exists $\alpha \in (0, 1)$ such that $L' \approx [\alpha (L), (1 - \alpha) L'']$, which implies $A^+ \cap A^- \neq \emptyset$. The uniqueness of α follows from the first part of the proof of the theorem due to I_R, and then $A^- = [0, \alpha]$ and $A^+ = [\alpha, 1]$ are closed sets. So, this fact finishes the proof.

4. Conclusions

As the main result of this paper, we show the equivalence among four textbook axiomatic systems regarding the expected utility theorem given by four authors. This implies that the objects satisfying the four axiomatic systems coincide. Also, we provide a common language for the space of lotteries, the simplification of a lottery, and their composition. These kinds of results could help the understanding of the logic and objects involved in different treatments of the expected utility theory.

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