R&D INVESTMENT IN STRATEGIC SETTINGS: A SURVEY OF PATENT RACES

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Resumen: Este trabajo revisa las principales contribuciones teóricas que estudian cómo las empresas invierten en **I** y **D** en un ambiente estratégico. Estudia el proceso de innovación desde el punto de vista de la oferta de nuevas tecnologías y revisa los modelos que racionalizan el proceso de innovación como una carrera estocástica entre empresas con el objetivo de obtener una patente.

Abstract: This paper surveys the main theoretical contributions that study how firms invest in R&D in a strategic setting. I study the innovation process from the point of view of the supply of new technologies and **I** review those models that rationalize the innovation process as a stochastic race between firms with the aim of obtaining a patent.

Introduction

There is ample evidence that shows that technological innovation has been a major source of economic growth in advanced economies. We also notice how technical advance improves the well being of citizens of modern industrial economies. We notice how cars become more efficient in fuel consumption and pollute less, how computers revolutionize the organization of modern corporations, how discoveries in medical equipment improve medical treatments for ill people, etc.

Despite all this evidence that highlights the importance of technological innovation, there has been little treatment of the microeconomic underpinnings of technological innovation in modern mainstream economic theory. However, as Dasgupta (1988) suggests: "Despite this long-term neglect in the main body of analytical economics there has grown... a large... literature on the economics of technological change" (Dasgupta, 1988).

For Schumpeter, probably the most prominent economist stressing the importance of technological advance, the reward of extraordinary profits stimulates entrepreneurs to innovate. To induce firms to undertake R&D, society must grant the innovator monopoly rights after achieving his goal (i.e. patent rights). However, even if patent rights are granted, "the opportunity to raise money for innovation and to realize a profit from it depends importantly on the economic environment in which it is realized" (Kamien and Schwartz, 1982, p. 1). The issue of the relation between market structure (the economic environment) and innovation was not properly studied until the early eighties with the adoption of the new techniques of game theory by the economics profession.[1] This paper surveys how firms invest in R&D in a strategic setting.

Specifically, I will review those models that rationalize the innovation process as a race between firms. The first firm to make the invention gets a patent, and obtains a prize R. As Kamien and Schwartz have stated, the innovation process can be analyzed from two different points of view; The supply of innovations (i.e. the creation of new technologies) and the demand for innovations (i.e. the adoption of new technologies). My main goal is to survey those models that analyze the innovation process from the point of view of the creation of new technologies. The aim of this paper is to present a unified approach of the economics of technological change from the point of view of patent races.

In this context, the literature has addressed a long list of issues: What is the role that the intensity of rivalry plays in the speed of development? What is the optimal number of projects? Does the

[1] Earlier studies were realized in the decision theoretic tradition, which assumed that the actions taken by one firm do not have an impact on the rival's decision.

market give as an outcome an efficient number of projects at the socially optimal level of effort? How do the policies of development change if the race is developed in stages? How does the competition for innovation change when we introduce a dynamic setting? What are the different incentives that motivate asymmetric firms to innovate?

These questions have been studied under a large variety of approaches which differ in the way the probability of discovery is modeled, the set of strategies that the models allowed, the number of stages in which the race is developed, the way the cost structure and the technology of innovation are specified and the role that time plays in the competitive setting. This paper surveys how the models differ in their specification of the last conditions, in what sense they are more or less restrictive and how the literature has modified them to try to explain properly how the innovation process works. Also, it compares the noncooperative Nash result with the social planner counterpart. Typically, the market outcome will yield excessive investment in R&D and too many firms devoted to R&D relative to the socially desirable level.

The paper presents a unified approach I start with the two canonical models by Loury (1979) and Lee and Wilde (1980), then I study how the literature has modified their basic settings to try to answer properly the questions posited above.

The survey is structured in the following way: In the first section, the seminal ideas of the game theoretic approach to Research and Development and market structure are studied. Important works in this area are the papers by Loury (1979), Lee and Wilde (1980) and Reinganum (1981, 1982). The models by Loury and Lee and Wilde constitute the basic paradigma upon which the issues of innovation and related issues have been studied. Also, they provide the simplest setup to analyze the issue of stochastic patent racing, even for asymmetric models. The differences between these models are addressed in this section. In particular, Loury and Lee and Wilde differ in the specification of the cost function. On the other hand, the models by Reinganum differ from those by Loury and Lee and Wilde in the specification of the probability distribution function of innovation. The reasons for this difference is that Reinganum intends to study a dynamic game.

The models surveyed in this section have the property that all their equilibria are of the precommitment type. The firms precommit either to a constant level of expenditure or to a path of expenditure on R&D that depends on time only.

In the second section, I survey the models that try to analyze the patent races in a truly dynamic game. These models focus their analysis on the actions and reactions that firms take as the race unfolds. This has been done by specifying multistage games and by transforming the functional form of the hazard rate (the probability that the firm will innovate in the next instant given that it has not innovated yet). Important works in this context are those analyzed by Fudenberg, Gilbert, Stiglitz and Tirole (1983) Grossman and Shapiro (1987), Harris and Vickers (1987), and finally Judd (1986). I also point out how open loop games can be transformed into a game with true feedback strategies by the use of asymptotic techniques. These techniques can be useful in dynamic models with nonexistent analytical solutions. For applications to growth models see Judd and Guu (1993).

Most of the literature surveyed in the paper focus on symmetric games. However, in the third section, I analyze more general settings in which firms face asymmetric incentives to innovate. For example, some of the firms may be currently producing a product that will be substituted by the innovation, whereas others are investing just enter into the market.

Finally, I analyze the efficiency properties of the market outcome. Several models are posited in a way in which the efficiency properties of the noncooperative Nash outcome can be studied. Efficiency can be analyzed in two dimensions. First, we can see whether the market allocations yield excessive allocation of resources to R&D, second, when there are several possible projects for R&D that vary in risk, we can see whether there is excessive allocation to risky projects.[2]

It turns out that most of the literature stresses the inefficiency of the market outcome. This result has been challenged by Sah and

[2] When there is a portfolio of research technologies, all aimed to the same innovation. The issue of correlation arises. We can check whether there is excessive correlation in the projects chosen by the market, when we compare the market outcome with the social planner solution. See Footnote 35 below.

Stiglitz (1987), arguing that in those models the firms are restricted to take just one project. If this constraint is relaxed, the social planner outcome and the market outcome coincides.[3] I point out in this part that the driving force of Sah and Stiglitz assertion is the assumption that states that more than one project can be successful. This result comes from the fact that they study an economy without time. If we reformulate the Sah and Stiglitz model into a game of timing in which the first to innovate takes all (i.e. a stochastic patent race), the market outcome is not optimal anymore.[4] Nonetheless, their intuition that the number of firms is not a relevant variable in determining the incentives for innovation turns out to be right in a noncooperative environment.

1. Seminal Works

The aim of this part is to analyze the seminal research papers that study the game theoretic approach to R&D: Loury (1979) and Lee and Wilde (1980). These models are important because they constitute the basic settings upon which the most recent literature builds its arguments. However, they are static games, firms choose their strategies in a Euclidean space and advantages in experience do not matter. At the end, I study the dynamization of these models by Reinganum (1981) with the use of differential games techniques. She sets up a model in which the rate of investment on R&D is in principle a function of the time elapsed and the state variables.

[3] If social benefits are equal to private benefits –i.e. if firms can perfectly price discriminate–, the former literature argued that the market outcome will still be inefficient. In contrast, Sah and Stiglitz argue that the only reason for the market outcome to be different from the social planner solution is the difference between private and social benefits.

[4] As far as I am aware, this model constitutes the first one to allow the firms to choose multiple uncorrelated projects, similarly to Sah and Stiglitz, that highlights the importance of the timing of innovation, and shows the inefficiency of the market outcome.

1.1. *Probability Distribution Function of Innovations*

The way in which the probability distribution function for innovations is specified is a salient feature that varies across these studies. Therefore, it is important to define it at the outset and to review the different ways in which this function has been modeled in the different papers. First of all, I must point out that most of the literature has posited a probability distribution function that guarantees solvability, or at least a simple characterization of the game.

Let us define $P_i(t)$ as the probability that the firm i has already innovated by time t. Then the probability that a firm will innovate in the next moment of time, given that it has not yet done so by time t (the hazard rate) is defined as:

$$g(t) = \frac{P'(t)}{1 - P(t)} \ . \qquad (1)$$

Where $P'(t)$ is the derivative of the distribution function.

Equation (1) represents a simple differential equation with solution equal to:

$$P(t) = 1 - \exp^{-\int_0^t g(v)dv} \qquad (2)$$

The literature has assumed different forms for the term $\int_0^t g(v)dv$. In particular, the models that $\int_0^t g(v)dv = \lambda t$. I will be reviewing in this section have assumed that. In this case the probability distribution function ($P(t)$) corresponds to an exponential distribution function with parameter λ. Consequently, from the definition of $g(t)$ (equation (1)), I get the following result:

$$g(t) = \lambda \qquad (3)$$

The different settings about market ant technical uncertainty can be modeled with the use of $g(t)$. For example, if we want to model market uncertainty, we define $g(t)$ as the firm's assessment of the conditional probability that the rivals will innovate in the next period of time, given that they have not done so yet. In this setting, the date of discovery is not uncertain, but the firms do not know the actions of their rivals. This specification has been used mainly in studies that approach the R&D process from the decision theoretic point of view, the maintained assumption of these studies being that firms consider that their choices do not affect their rivals' decisions.[5]

On the other hand, even if firms have perfect information on their rival's strategies, they may still face uncertainty because the date of discovery is itself uncertain (technical uncertainty). In this case, the function represents the conditional probability for a representative firm that it will innovate in the next moment of time given that it has not done so. So, usually the function $g(t)$ has been used in the game theoretic literature to represent the realistic assumption that innovation is itself an uncertain outcome.[6] In this work, I discuss only models that entail technical uncertainty.

It is meaningful to stress that the existence of technical uncertainty is necessary for the existence of competition for innovation in a game of timing. If firms were aware of the date of discovery, all firms would spend to the point in which the cost of innovation is equal to the prize of obtaining it. However, since all firms will innovate for sure, the prize will be shared by all. A symmetric pure strategy equilibrium will not exist in this context (Dasgupta and Stiglitz, 1980).

The models I discuss in this paper have posited different forms of the function $g(t)$. In particular, Loury (1979) and Lee and Wilde (1980) assume that depends upon the strategies chosen by the firms at the beginning of the race.

On the other hand, Reinganum (1981, 1982) assumes that is equal to a constant times a function that depends on the rate of knowledge acquisition chosen by the firm at that moment in time. Because Reinganum uses differential game techniques with terminal dates, she will obtain a path of R&D expenditure that is not constant over time.

In the second section I review the model by Fudenberg, Gilbert, Stiglitz and Tirole (1983). In their study, the function $g(t)$ is equal to a function of the level of experience accumulated through time. In the variation on Reinganum's model, I make $g(t)$ a combination of the

[5] See Kamien and Schwartz (1982).

[6] I should mention that the function $g(t)$ may still represent market uncertainty in a game theoretic framework. In this case, although firms are behaving strategically, they do not have perfect information on their rival's strategies.

Fudenberg *et al.* paper and the Reinganum (1981) model. Like Fudenberg *et al.*, I make a function of the accumulated level of experience. However, $g(t)$ also depends on the rate of knowledge acquisition chosen at that moment in time, similarly to Reinganum.[7] The rationality of these approaches will become clearer later in the paper. Here, I only wanted to stress how important the different specifications are.

1.2. *Model with Lump Sum Costs*

Consider the following assumptions:

1.2.1 The probability of discovery is independent across firms, i.e. the projects of R&D undertaken by any single firm are uncorrelated with those chosen by other firms (i.e. there are no externalities in R&D).

1.2.2 The firms face technical uncertainty.

1.2.3 The probability that a firm has already innovated by time t follows an exponential distribution function, with parameter λ.

1.2.4 The function $g(t)$ depends on the level of outlays made by the firm.

1.2.5 The outlays are made in lump fashion (all at once).

1.2.6 Each firm undertakes only one project.

1.2.7 The firms are competing for a fixed reward of size R (a fixed sum) and the loser gets nothing.

Assumptions 1.2.1, 1.2.2 and 1.2.7 will hold for almost the whole paper. Assumption 1.2.7 implies that no firm is currently producing a product that will be supplanted by the innovation, and that all the resources allocated to R&D are lost if the firm looses. In section 3 I will relax this assumption. Assumption 1.2.5 describes a situation in which technological development involves mainly fixed costs, and it will be relaxed in the next section. Assumption 1.2.6 will be modified in section 4.2 and 4.3. Assumption 1.2.4 will be changed continuously, given that the literature differs more in this assumption.

[7] Judd (1986) has a similar setting for his risky projects.

Let us analyze the problem from the point of view of firm i. Under the above assumptions, the probability that the rivals of firm i have not innovated by time t is given by the next expression:

$$\prod_{j \neq i}(1 - P_j(t)) = \prod_{j \neq i} e^{-\lambda(c_j)t}$$

where c_j is the level of expenditure chosen by i's rivals. Consequently, the probability that the rivals of firm i will have innovated by time t is $1 - \exp^{-at}$, where $a = \sum_{j \neq i}^{n} \lambda(c_j)$. Intuitively, I can define $P'(t) = ae^{-at}$ as the probability that any of the rivals innovate at time t. By the same token, $\lambda e^{-\lambda w}$ is the probability that firm i innovates at time w.

Firm i will have the prize R only if it innovates before its rivals do. Therefore, expected profits for firm i are:

$$V_i(R, r, a) = \max_{c_i} \int_0^{\infty} ae^{-at}[\int_0^t \lambda(c_i)e^{-\lambda(c_i)w} Re^{-rw} dw]dt - c_i$$

The above equation states that if any of the rivals of firm i innovate at time t, then for firm i to win the prize R, it has to innovate before that time. The inner integral calculates that possibility. The other integral calculates the possibility that any of the rivals of firm i innovate by time t. By simplifying the above expression:

$$V_i(R, r, a) = \operatorname*{Max}_{c_i}[\frac{R\lambda(c_i)}{a + \lambda(c_i) + r} - c_i] \qquad (3')$$

Before I characterize the first order conditions, I must make some assumptions about the technology of innovation.

1.2.8 I assume that and $\lambda'(c) > 0$ and $\lambda''(c) < 0 \;\forall\; c$ s.t. $c > 0$, $\lim_{c \to \infty} \lambda'(c) = 0$ and $\lim_{c \to 0} \lambda'(c) = \infty$.

The last assumption simply states that if we double the amount of expenditure, the conditional probability of success (λ) will not double. The reason for this result may be the existence of some fixed factors in the research program (human capital for example). The assumption also guarantees the existence of a Nash equilibrium.[8]

[8] See Harris and Vickers (1987).

Intuitively, it is a sufficient condition that guarantees the interiority of the optimal response function.

The necessary first order conditions of the firm optimization problem are:

$$\frac{(a+r)R(\lambda'(\hat{c}_i))}{(a+\lambda(\hat{c}_i)+r)^2}=1 \tag{4}$$

The sufficient second order conditions are:

$$\frac{(a+r)R[\lambda''(\hat{c}_i)(a+\lambda(\hat{c}_i)+r)-2\lambda'(\hat{c}_i)]}{(a+\lambda(c_i)+r)^3}<0$$

To characterize the game theoretic solution it is necessary to find the sign of the response function to a change in the parameter a (the hazard rate). Differentiating (4) with respect to a and taking into account the symmetry of the Nash Equilibrium:

$$\frac{\partial\hat{c}_i}{\partial a}=\frac{\lambda'(\hat{c}_i)[(n-2)\lambda(\hat{c}_i)+r]}{(a+r)R[\lambda''(\hat{c}_i)(a+r+\lambda(\hat{c}_i))-2\lambda'(\hat{c}_i)^2]}<0 \tag{5}$$

Given the symmetric nature of the game, an increase in rivalry is defined as an increase in the number of firms competing for the reward. From equation (4) and the symmetry of the Nash Equilibrium:

$$\hat{c}_i+\hat{c}_i((n-1)\hat{c}_i,r,R) \tag{6}$$

Differentiating equation (6) and using (5) we find that $\frac{\partial\hat{c}_i}{\partial n}<0$. This means that an increase in rivalry reduces the level of expenditure by a representative firm. Further, by differentiating (4) and using (5) and (6) we find that $\frac{\partial\hat{c}_i}{\partial R}<0$. The last argument can be summarized in the following proposition.

PROPOSITION 1 (Loury, 1979). *With assumptions 1.2.1-1.2.8, we have the following results:*

1. An increase in rivalry reduces the effort of each firm.

2. An increase in the prize of innovating has a positive effect in the effort of each firm.

The intuition from the last proposition is as follows: An increase in the number of firms reduces the expected date of discovery, increasing at the same time the likelihood of loosing the race. This result reduces expected profits, forcing firms to reduce the rate of effort. An increase in the prize for winning R, increases expected profits, thereby inducing firms to invest more in R&D.

1.3. *Model with Flow Costs*

It seems excessive to assume that the costs of R&D are committed at the beginning of the race. Instead, it is reasonable to argue that firms make their expenditures on R&D only as long as the race lasts. So, let us assume that we have two different types of costs. First, we have a fixed cost that is paid by the firm in order to enter the race. The second is a flow cost, spent by the firm only as the race lasts. The conditional probability of innovating in the next moment of time, given that it has not innovated yet (λ), is a function of the flow cost (c). In this framework, the total amount of flow costs incurred by the firm depend on t, the date at which the race is over:

$$c(t) = \int_0^t ce^{-r\tau}d\tau$$

To obtain the expected costs we must weight $c(t)$ by the probability that the race is over at each moment t.

$$P'(t) = (a + \lambda)e^{-(a + \lambda t)}.$$

Integrating over the relevant range we get the expected costs:

$$EC = \int_0^{\infty}(\lambda + a)e^{-(\lambda + a)t}[\int_0^t (c)e^{-r\tau}d\tau]dt = \frac{c}{a + \lambda + r}$$

Therefore, under the assumption of flow costs,[9] expected profits for firm i are given by the following expression:

[9] Kamien and Schwartz name this case the "noncontractual cost case".

$$V_i^*(R, r, F, a) = \underset{c_i}{\text{Max}}[\frac{R\lambda(c_i) - c_i}{a + \lambda(c_i) + r} - F] \qquad (7)$$

Calculating the necessary first order conditions and using (7):

$$\lambda'(c_i^*) = \frac{1}{R - (V_i^* + F)} \qquad (8)$$

Using equation (8) we can conclude the following: If a goes up then (7) indicates that V_i^* goes down, by the concavity of $\lambda(c_i)$ and to satisfy equation (8), c_i must go up. Therefore, $\frac{\partial c_i^*}{\partial a} > 0$. By differentiating c_i with respect to n (the number of firms) and imposing the symmetry condition:

$$\frac{\partial c_i^*}{\partial n} = \frac{\frac{\partial c_i^*}{\partial a}\lambda(c_i^*)}{[1 - \frac{\partial c_i}{\partial a}(n-1)\lambda'(c_i^*)]} \qquad (9)$$

If we assume that $\frac{\partial c_i}{\partial a}\lambda'(c_i^*)(n-1) < 1$,[10] the last equation is positive. Similarly, by differentiating the first order conditions it can be easily checked that $\frac{\partial c_i}{\partial R} > 0$.

PROPOSITION 2 (Lee and Wilde, 1980). *Under assumptions 1.2.1-1.2.4 and 1.2.6-1.2.8, a cost function that emphasizes the duration of the race as determinant of the total costs, and if* $\frac{\partial c_i}{\partial a}h'(c_i^*)(n-1) < 1$, *the following conditions hold:*

1. As the degree of rivalry increases, the representative firm will increase its outlays in Research and Development.

2. If the size of the reward goes up, then all firms will increase their R&D efforts.

[10] This is a stability condition. If the rivals of any firm increase their investment in such a way that rivalry increases in one unit, then the firm should increase its investment in less than one unit. For the two firm case, it implies a reaction function with a slope less than one.

The contrasting results between propositions one and two deserve some comments. First, the striking difference in the slope of the reaction function is entirely due to the specification of the cost function. In the first proposition, outlays are made once and for all. In contrast, in the second, expenditures depend on the duration of the race. In both models an increase in rivalry reduces expected profits of the representative firm (V and V^*), but in the second case, costs go down too. This latter assertion follows from the noncontractual property of the cost function.

The second consideration is related to the efficiency properties of the models. As we will see later in this paper, both models yield similar efficiency conclusions with regard to the comparison between the socially optimal number of firms and the number given by the market. However, the level of effort is excessive for the noncontractual model when we compare it with the socially optimal level. For the contractual cost model, the market level of effort is socially insufficient. The choice between the two models depends upon the problem we are trying to study. If we have a strong suspicion that the R&D process involves mainly fixed costs (i.e. the construction of labs, the acquisition of sophisticated equipment, etc.), the first model is the most representative.

The third consideration is related to the implications of these static models in the context of multistage races. As we will see later in the paper, the results in multistage races depend heavily on the sign of the response function.

The fourth consideration deals with empirical implications of these models. If we assume that all the firms competing for the innovation coexist in a symmetric industry, then the prediction of the first model implies that as the degree of concentration increases, firms will spend more on R&D. The second model would predict the opposite.[11] Scherer (1967) concluded that the hypothesis of positive association between concentration and R&D was supported, thereby confirming the predictions of the first model. In Castañeda (1994 in progress) it is

[11] I am assuming that the degree of concentration is measured by the Herfindahl Index. In the case of symmetric firms, the Herfindahl Index is given by the reciprocal of the number of firms.

found that market power in México is positively associated with the rate of growth of total factor productivity.[12]

1.4. *A Differential Game Approach*

The last two propositions were obtained under rather static assumptions: 1) The experience accumulated by firms through time did not play any role in the probability of success. In other words, the effects of the process of learning by firms are ruled out. 2) The strategy space is a subset of a euclidean space, firms choose their strategies once and for all at the outset, the rate of expenditure is restricted to be constant over time.

The work done by Reinganum (1981, 1982) attempts to improve upon the last two problems. In order to do that, Reinganum changes the specification of the function $P_i(t)$. Instead, she proposes the next form:

$$P_i(t) = 1 - e^{-\lambda k_i(t)} \tag{10}$$

As before, $P_i(t)$ represents the probability that firm i has innovated by time t. Notice that in this setting, this probability is a function of $k_i(t)$, the level of knowledge accumulated through time. The probability of success in the next moment, given that it has not so done yet, is:

$$g_i(t) = \lambda k_i(t)$$

where $k_i(t) = \dot{k}_i(t)$ (i.e. the rate of knowledge acquisition chosen for that moment of time). The last equation shows us how Reinganum's model relates with the models analyzed earlier in this paper. As in the earliest models "the exponential distribution shows that each increment in knowledge is equally likely to be the one which provides success" (Reinganum, 1981, p. 24). Because Reinganum uses differential game techniques with terminal dates, she distinguishes from the former models in allowing a level of effort that, in principle, depends

[12] The rate of total factor productivity is considered as an index that reflects innovative effort. There is no data of R&D for México.

on state variables and time. To solve the model, Reinganum makes several additional assumptions.

1.4.1 If the firm has not succeeded in developing the new product after a certain period of time T, it will leave the race.

1.4.2 The cost of acquiring additional knowledge is

$$c_i(k_i) = 1/2(k_i)^2.$$

1.4.3 The rate of knowledge acquisition per unit of time is bounded.

1.4.4 (Lifshitz properties) The strategies $t_i(t, k)$, are bounded and continuous t and k. Further if k and $\bar{k}$ are close, then r_i and $\bar{r}_i$ are close too.

1.4.5 Costs are discounted but the prize is not.

I maintain assumptions 1.2.1, 1.2.2, 1.2.6 and 1.2.7 of section 1.2. Notice that assumption 1.4.2 implies that costs are noncontractual. Firms will spend on R&D only as long as the race continues. On the other hand, it is interesting to notice the restrictiveness of assumption 1.4.4. As Fudenberg and Tirole (1986) have pointed out, the continuity of strategies upon the state in a differential game is made in order to find a solution. The assumption of continuity has not been justified on other grounds by the literature. Finally, 1.4.5 is used to simplify the algebraic calculations. In particular, she uses this assumption to integrate by parts an expression. I will obviate the mathematical developments of the Reinganum's model because it constitutes a particular case on the most general case considered below. The following proposition summarizes her accomplishments.

PROPOSITION 3 (Reinganum, 1982). *Under assumptions 1.4.1-1.4.5, 1.2.1, 1.2.2, 1.2.6, 1.2.7 and equation (10) we get the following results:*

1. The strategies are open loop.

2. As time passes, the rate of knowledge acquisition goes up.

3. As R goes up, the rate of knowledge acquisition goes up.

4. As T increases, the rate of knowledge acquisition goes down.

5. As the intensity of rivalry goes up the rate of knowledge acquisition goes up.

The major achievement of Reinganum's model is conclusion 2. The result follows from assumptions 1.4.1 and 1.4.5. If these assumptions are modified, i.e. we discount prize as well as costs and there is no terminal date, the firms will not vary their efforts with time. "Indeed the analysis of Reinganum's modified model shows that, even when the strategy spaces of Lee and Wilde's model are enlarged, equilibrium remains the same as the originally determined by them... this provides some justification for employing the simpler strategy space" (Harris and Vickers, 1987, p. 2) of Lee and Wilde. Conclusion 1 shows us that the model of Reinganum was not very useful in helping us to understand real dynamic interactions in which the decisions of a firm depend upon the progress made by the rivals and itself. The closed loop strategies coincide with the open loop strategies. This follows from the memorylessness property of the exponential distribution function. In this sense, Fudenberg and Tirole (1986) call this model static.

The models that I survey in the next section try to overcome the open loop feature of the models considered so far.

2 Dynamic Models

In the models studied so far, firms have had the same probability of winning the race if all were to invest at the same level. This fact originates from the memorylessness property of the exponential distribution function.

To overcome this problem, the models in this section define state variables that give the firms different possibilities for winning the race, even if they have the same level of investment in R&D at that moment in time. The first way of introducing state variables in these models has been to allow the parameter to depend upon the level of experience accumulated by the firms. Another way of introducing state variables is by studying a race that entails more than one stage, a firm will be in a later stage, only if it has succeeded in earlier stages. In contrast with Reinganum, in all models in this section, the equilibrium strategies depend on state variables. This fact allows us to study the reactions of firms to changes in the conditions of competition as the race develops. Firms make their decisions depending on the progress that the rival has made and view how their decisions may affect the rival's choices tomorrow.

The literature has differed in its approach to dynamizing the models. For example, the work by Fudenberg, Gilbert, Stiglitz and Tirole (1983) has used both ways of introducing state variables. On the other hand, the work by Grossman and Shapiro (1987) has studied a two-stage race.

The work by Judd (1986) has analyzed a multistage race and has made the function $g(t)$ dependent on the position of the firm in the race as well as the level of outlays allocated to R&D. He also allows for another type of less risky projects that allow the firm to improve its position and to reach states closer to the end of the race. However, because Judd uses asymptotic methods, he has to compromise by assuming that the prize of success is arbitrarily small.

Finally, similarly to Judd (1986), a variation on Reinganum's model assumes that $g(t)$ is a function of the level of experience accumulated and the rate of knowledge acquisition chosen by the firm at that moment in time. This model constitutes a hybrid between the Fudenberg *et al.* (1983) model and the Reinganum (1981, 1982) model.

2.1. *Changing Hazard Rate and Restrictive Set of Strategies*

Suppose now that the parameter λ is a function of the level of experience accumulated by the firm. Consequently, the function $g_i(t)$ has the following form:

$$g_i(t) = \lambda(k_i(t))$$

where $k_i(t)$ is the level of experience accumulated by firm i. From:

$$g_i(t) = \frac{P_i'(t)}{1 - P_i(t)}$$

We have that:

$$P_i(t) = 1 - e^{\int_0^t \lambda(k_i(s))ds}$$

Assumption 2.1.1: There are just two firms competing for the prize.

The probability of discovering at time t is

$$\lambda(k_i(t))e^{-\int_0^t \lambda(k_i(s))ds},$$

where $k_i(t)$ is the level of experience accumulated at time t. Expected profits at time t are:

$$\Pi_i(t) = e^{-\int_0^t (\lambda(k_j(s)) + \lambda(k_i(s))ds}[\lambda(k_i(t))R - C_i] \tag{16}$$

Calculating the present value of the last expression we get the following expression for expected profits:

$$\hat{\Pi}_i = \int_{t_i}^{\infty} e^{-rt}[e^{-\int_0^t (\lambda(k_j(s)) + \lambda(k_i(s)))ds}]\,[\lambda(k_i(t))R - C_i]dt \tag{17}$$

where t_i is the period in which firm i joins to the race.

By looking at equation (16) we can notice the difference between this model and the one by Reinganum analyzed in the last section. In that model, the function $\lambda(k_i(t))$ was equal to a constant λ for all times t. All firms had the same level of expected benefits regardless of the level of experience. In contrast, in this model, if firm i has a higher level of experience: $\Pi_i > \Pi_j$.

In addition to the new specification of the function $g_{(t)}$, consider the following assumptions:

2.1.2 It is not feasible for both firms always to engage in R&D.

2.1.3 It is feasible for one firm to do R&D.

2.1.4 The level of effort chosen by each firm is restricted to be the same at every moment of time. (The strategies are restricted either to do R&D or not to do it. When the firms do R&D, they can chose only one level of effort.)

I maintain assumptions 1.2.1, 1.2.2, 1.2.6 and 1.2.7.

PROPOSITION 4. *In a patent race with the last assumptions and in which the function $g(t)$ is dependent upon the level of experience: A small advantage in the level of experience by any firm at the start of the race will result in a subgame perfect equilibrium in which the firm with the highest level of experience does R&D and the other leaves the race at the start.*

Intuition of the proof:[13] By assumption 2.1.2, equation (17) is negative for firm i at time zero. But assumption 2.1.3 implies that there exists a level of experience $\bar{k}$ such that equation (17) is positive. Under assumption 2.1.4, the firm with the highest level of experience will reach that level before the other firm does. When this happens, the firm with the lower level of experience has equation (17) still negative, but the firm with the highest level of experience has a dominant strategy to continue. Besides, assumption 2.1.2 implies that both firms cannot continue the race forever. By backward induction, the firm with the lower level of experience will leave the race at the start.

As Fudenberg *et al.* have pointed out, the last proposition is true regardless of the stochastic nature of the game. Notwithstanding the fact that discovery is uncertain and the firm with the lower level of experience may discover before, it is not optimal for it to undertake in R&D. Proposition 4 is the stochastic version of the Dasgupta and Stiglitz (1980) result mentioned above. In that model, the date of discovery is certain and due to the argument advanced in 1.1, symmetric pure strategy Nash equilibria does not exist. Therefore, for a solution to exist in pure strategies, we must focus our study on asymmetric models. The firm with some kind of advantage will be the only one that does R&D.

The key point is that the leader can reach a point in which he can guarantee to himself a positive level of profits before the follower does. We may notice that in spite of the fact that in principle, the last model allows the firms to take different positions in the race, it is not useful to analyze dynamics because the only possible actions allowed are restricted to doing R&D at a constant level or exiting the race.

Lippman and McCardle (1988) change the implicit assumption of the last model which allows firms to decide at each instant in continuous time about continuing the race or dropping. Lippman and McCardle argue that if we restrict the decision period to a certain amount of time $t > 0$, so that firms can decide to invest or drop out only at $(1t, 2t, 3t, \dots)$, the ε preemption result is not the unique equilibrium.

[13] This intuition follows Fudenberg, Gilbert, Stiglitz and Tirole (1986), see the authors for the formal proof.

Indeed, Lippman and McCardle show that there may be three possible subgame perfect equilibria: The original equilibrium in which the follower drops out at the start, a second equilibrium in which the leader drops out in the second period of decision, and a third equilibrium which has a mixed strategies solution. In this latter equilibrium, the leader has a higher probability of dropping out. The reason is that continuing is more valuable for the leader. To satisfy the indifference condition of mixed strategies, the follower must have a higher probability of continuing.

Harris and Vickers (1985) analyzed the preemption result in asymmetric models. However, because the asymmetric model implies harder conditions in the analysis, they compromise by studying a model without uncertainty (see section 4 below).

Another way to overcome the preemption result and analyze dynamic interactions is to introduce stages. This will allow for the possibility of one firm to advance and for the other to draw even, or to "leapfrog".

As a simple example,[14] consider a two stage game in which the function g is dependent upon the level of experience achieved by the firm and the assumptions stated in this section are true. The stages are symmetric. In both stages, $g(t)$ is dependent upon the level of experience. In this setting, a higher level of experience by any firm does not preclude the other from doing R&D. In this case, the firm with the lower level of experience can make the preliminary discovery first and advance in its position relative to the more experienced firm. Once one of the firms has succeeded in the first stage, the rival will abandon immediately. From this point of time on, we have exactly the same game as in the last proposition. As the last example shows, firms may have different positions and still win the race. Moreover, the leader may loose the race. The key point that allows dynamics in the game is the introduction of stages in the race. If the stages are not treated symmetrically, we might get several results depending upon the specification of the model. Fudenberg *et al.* (1983) have studied these cases. The aim of those models is to illustrate the fact

[14] See Fudenberg, Gilbert *et al.* (1983).

that when we introduce both the multistage feature and the level of experience of the firms as relevant variables, firms will have different relative positions and the intensity of competition will depend on how these two state variables interact. To understand multistage races, we must continue with the next sections.

Another possibility is to change the assumptions of the model, the preemption result depends heavily on the fact that firms can do R&D only at a constant rate and the specification of the function $g(t)$. Later we will see that a variation on Reinganum's model allows us to have a single stage and we do not have ε preemption.

Lippman and McCardle (1987) study a multistage race similar to Fudenberg *et al.* setting. They also restricted the strategies to two possibilities: Either doing R&D at a given level or dropping the race. They show that as long as the firms have the same level of success, both firms do R&D. Once one of the firms reaches an important lead, the follower abandons the race.

2.2. *Multistage Races, Unrestricted Strategy Spaces and Constant Hazard Rate*

The work by Grossman and Shapiro (1987) relaxed the assumption of a constant rate of effort. The game is exactly as the one studied in section 1.3, the only difference is that two firms will play the game in two stages. So, assumptions 1.2.1-1.2.4 and 1.2.6-1.2.8 from section 1.2 are still valid. Besides, costs are noncontractual as in section 1.3. Since we are in a multistage game, we must say something about the structure of information: Firms are in a perfect information environment, and can observe perfectly the four possible states of the world: 1) Both firms are in the first stage. 2) Both are in the second. 3) One firm is in the advanced stage and the other is in the first stage. 4) The roles in 3 are reversed. Since expected payoffs depend upon the state of the game, the strategies will vary accordingly. Therefore we have a truly dynamic game.

Denote by V_{LL} as the expected payoff that a typical firm faces if both firms have succeeded in the first stage. Define V_{LF} as expected profits for a firm that has succeeded in the first stage, but its rival has not. By the same token V_{FL} represents expected profits for a firm that has not passed the first stage but its competitor has. Finally, V_{FF}

represents expected profits of a firm in the situation in which none of them has succeeded in passing the first stage. The subindices for the amount allocated to R&D, c, have the same meaning.

By equation (7) in section 1.3:

$$V_{LL}(c_r, R, r) = \underset{c_{LL}}{\text{Max}}\left[\frac{R\lambda(c_{LL}) - c_{LL}}{r + \lambda(c_{LL}) + \lambda(c_r)}\right] \tag{18}$$

Where c_r corresponds to the level of expenditure chosen by the rival. We can get the mathematical representation of V_{FL}, V_{LF} and V_{FF}, by noting that the first term in equation (18), represents the expected reward given the state of the game (both firms are in the second stage). Following the same reasoning, we can get expected payoffs for the firm given each possible state of the game.

$$V_{FF}(V_{LF}, V_{FL}, r, c_r) = \underset{c_{FF}}{\text{Max}}\left[\frac{V_{LF}\lambda(C_{FF}) + V_{FL}\lambda(c_r) - c_{FF}}{r + \lambda(c_{FF}) + \lambda(c_r)}\right] \tag{19}$$

$$V_{LF}(R, V_{LL}, r, c_r) = \underset{c_{LF}}{\text{Max}}\left[\frac{\lambda(c_{LF})R + \lambda(c_r)V_{LL} - c_{LF}}{r + \lambda(c_{LF}) + \lambda(c_r)}\right] \tag{20}$$

$$V_{FL}(V_{LL}, c_r, r) = \underset{c_{FL}}{\text{Max}}\left[\frac{\lambda(c_{FL})V_{LL} - c_{FL}}{r + \lambda(c_{FL}) + \lambda(c_r)}\right] \tag{21}$$

The corresponding first order conditions can be simplified to:

$$\lambda'(c_{LL}) = \frac{1}{R - V_{LL}} \tag{22}$$

$$\lambda'(c_{FF}) = \frac{1}{V_{LF} - V_{FF}} \tag{23}$$

$$\lambda'(c_{LF}) = \frac{1}{R - V_{LF}} \tag{24}$$

$$\lambda'(c_{FL}) = \frac{1}{V_{LL} - V_{FL}} \tag{25}$$

We can use (18) to (25) to prove the following proposition:

PROPOSITION 5 (Grossman and Shapiro, 1987). *In a two stage innovation race with assumptions 1.2.1-1.2.4 and 1.2.6-1.2.8 –stated in section 1.2– the assumption of noncontractual costs and perfect information, the following results hold:*

1. Both the leader and the follower speed up should the race become tied.

2. The intensity of rivalry is higher in the second stage than in the first one.

3. The leader always spend more than the follower.

I omit the proof. A key element in the proof is the positive response function generated by the noncontractual cost.

The first result "reflects the fact that the leader has more to lose and the follower has more to win" (Grossman and Shapiro, 1987, p. 377) should the race become tied. The intuition behind the second result is given by the positive discount rate, which forces both firms to spend more when they are closer to the end. The fact that the response function is positive as in section 1.3, reinforces this effect. This result is reminiscent of conclusion two in proposition three above.

It is not possible to compare the relation between c_{LF} and c_{FF} (as well as the relation between c_{FL} and c_{FF}). The reasons arise from two contradictory effects. On one hand, the fact that the discount factor is positive makes the leader spend more compared to what he should spend if the race were still in the first stage for both firms. But on the other hand, the fact that the optimal response function is positive and the follower may spend less forces the leader to spend less. Alternatively the follower may spend less because $V_{FL} < V_{FF}$, but the fact that the leader has an incentive to spend more (due to the discount factor) and the positive response function, gives the follower an incentive to spend more. A symmetric explanation holds for the second undeterminate relation.

If costs are contractual, we should be able to sign the relation between c_{FF} and c_{LF}. For this case, the response function is negatively sloped. So, the discount factor makes the leader spend more compared to what he should spend if both were in the first stage. If the follower reduces his effort, his impact on the leader's decision further enhances the discount factor effect. This follows from the negative response function. The impact on the follower is consistent with the last reasoning, the reduction in expected profits ($V_{FL} < V_{FF}$) makes the follower spend less in R&D. The fact that the discount factor effect makes the leader spend more and the negative response function further enhance this effect. However, for the contractual case, point two in proposition 5 will no longer hold. The positive discount factor makes both firms spend more, but the negativity of the response function goes in the opposite direction, leaving undeterminate the final effect. As in section 1, this reasoning illustrates the dependence of the results on the specification of the cost function.

2.3. *Asymptotic Methods*

2.3.1. Unrestricted Strategy Spaces, Multiple Stages and Multiple Feasible Projects

The model studied in section 2.1 severely restricted the strategy space. The model in section 2.2 studied a two-stage game. Both models illustrate the difficulty of analyzing dynamic Markov games unless restrictive assumptions are stated (restrictive strategy space, or few stages). Kenneth Judd (1986) proposes a novel approach to analyze patent races which circumvents the restrictiveness of the models in the last two sections. He uses perturbation analysis to study general dynamic multistage patent races with unrestricted strategies. He posits a probability of discovery (of finishing all stages at once) that depends upon the stage in which the firm is located and upon the amount of resources allocated to that project. He also assumes that there exists another class of projects which allow the firm to make gradual jumps from one stage to another closer to the end, or may be to the end. These two projects for doing R&D vary in their riskiness, with the projects that allow the firms to leap to the end being riskier.

The general strategy for asymptotic methods is to find a known solution, then use that solution for starting points to calculate more interesting nearby problems. Usually, the known solution has less theoretical interest. Asymptotic theory[15] uses a generalization of the implicit function theorem and Taylor series to functional spaces. The procedure is to calculate a Taylor series expansion evaluated at

[15] The reader interested in asymptotic methods should consult Judd (1992).

the known solution for the value function and the policy function. Judd uses these techniques to analyze closed loop solutions of games. Of course, there is a tradeoff in using asymptotic methods: He first solves for the case of zero prize for the patent race, an uninteresting problem, then he assumes a *small prize* in order to characterize the closed loop solution.

Let me maintain assumptions 1.2.1, 1.2.2 and 1.2.7, and assumption 2.1.1. Also, R is very close to zero.

Let $V^i(K_i, K_j)$ be the value function for firm i when it is in stage K_i and its competitor is in stage K_j and both players choose feedback Nash strategies. Let r_s^i be the allocation of expenditure to the safe project and r_r^i the level of expenditure allocated to the risky project. Like Reinganum (assumption 1.4.2), Judd assumes quadratic costs in each one of the projects. The riskier projects allow the firm to leap to the end of the race and the probability of this event is $\lambda(K_i)r_r^i dt$.

As in the model studied in the next section, the probability of finishing the race is a function of the position now and the amount of investment on R&D. On the other hand, the less risky projects allow the firm to jump to states either closer to the end or to the end. The probability that a partial jump occurs is $r_s^i dt$, and the probability of a jump from state K_i to state $(g, g + dg)$ if a partial jump happens is $\beta(g, K_i)dg$. On the other hand, the probability that a partial jump hits the end is equal to $r_s^i \mathrm{B}(K_i)$. By using the infinitesimal notation, the Bellman equation can be written in the following intuitive way:

$$
\begin{aligned}
V^i(K_i, K_j) = \underset{r_r^i r_s^i}{\text{Max}}\ [&-\frac{1}{2}r_r^i - \frac{1}{2}r_s^i \\
&+ r_r^i\lambda(K_i)Rdt(1 - rdt) + r_s^i dt(1 - rdt)\left(\int_{K_i}^{E} V^i(m, K_j)\beta(m, K_i)dm\right) \\
&+ r_s^i dt(1 - rdt)R\mathrm{B}(K_i) + r_s^j dt\left(\int_{K_j}^{E} V^i(K_i, g)\beta(g, K_j)\,dg(1 - rdt)\right. \\
&+ (1 - rdt)(1 - (r_s^i - r_s^j - r_r^i\lambda(K_i) - r_r^j\lambda(K_j))dtV^i(K_i, K_j))]
\end{aligned}
\tag{26}
$$

The explanation of the above expression is straightforward. The firm spends $-\frac{1}{2}r_r^i - \frac{1}{2}r_s^j$, with probability $r_r^i\lambda(H_i)$, it will leap to the end

(*E*) and get R in the next instant. The next term represents the possibility that firm i jumps to any state between K_i and the end (*E*). A symmetric explanation holds for the next term, this time firm j is the jumping firm. Finally, the last term represents the possibility that none of the firms succeed in having either a partial jump or a leap, so that in the next instant both firms are in the same state (K_i, K_j).

For a neighborhood of 0, $(0, R)$, the value function can be expressed in the following way:

$$V^i(t, K_i, K_j; R) \simeq V^i(t, K_i, K_j, 0) + RD^1(t, K_i, K_j; 0) + (R)^2 D^2(t, K_i, K_j; 0) + \ldots \qquad (27)$$

where

$$D^i(t, K_i, K_j; 0) = (\frac{1}{n!}) \frac{\partial^n V^i(t, K_i, K_j; 0)}{\partial R^n} \qquad n = \mathrm{i}, 2, \ldots$$

To find the coefficients of the R terms in (27), it is necessary to follow the following steps: From the first order conditions we plug the results for r_s^i, r_r^i and r_j^s in (26). The next step is to plug (27) on both sides of (26), then we equate the terms linear in R to get the coefficient of R in (27). We proceed by using this solution and equate the terms that multiply R^2 on both sides of (26) and continue in this fashion. Then we plug this solution in the first order conditions for the policy functions: r_r^i, r_s^i, r_r^j and r_s^j. Following this procedure, Judd was able to prove the following proposition:

PROPOSITION 6.

1. The optimal response of firm to an improvement in the position of firm is to increase its allocation to risky projects, but this is socially undesirable. For the safe projects, the response is undeterminate.

2. The leader spends more in both partial jumps and leaps.

The conclusions of the last proposition are consistent with proposition 5. Because Judd uses asymptotic methods he can study more general settings (multistage races with varying degrees of riskiness in projects) which give rise to conclusion 1 of the last proposition.

The discussion in this paper assumes there is no separation between ownership and management. If managers are not the owners of the firms, profit maximizing may not be the goal of the managers. In this case, Holmström (1983), and Holmström and Ricart Costa (1986) have shown in another context that managers will choose the less risky projects when considering different investment projects with varying degree of risk. If we translate their result to this survey, it implies that the separation of property and control will make firms choose less risky projects of R&D.

Harris and Vickers (1987) analyze a multistage version of the Beath, Katsoulakos and Ulph (1989) model which I review below. They study the case with no discounting. They consider two models: In the first model, the race is won whenever the leader achieves a given lead over his rival. The state corresponds to the difference between the two firms. They show that the leader makes greater efforts than the follower. If the difference between the two players increases, the follower diminishes his efforts. Similarly to Judd and Grossman and Shapiro, their second model is a two state variable model. The winner is the first firm to achieve a certain number of stages. If the leader does not have more than two stages to go, he spends more in R&D. Also, when the leader has no more than two stages to go, both firms, the leader and the follower, speed up when the gap between the two diminishes.

As shown in proposition 5, Grossman and Shapiro obtain similar results in a simpler model. Indeed, this shows that when the number of stages is enlarged, the main qualitative features that we get from two-stage models remain valid. Furthermore, Harris and Vickers could not obtain additional results. This undermines their accomplishments and provides justification for restricting attention to simpler models such as the one by Grossman and Shapiro. On the other hand, the last proposition shows that asymptotic methods give more qualitative results. Therefore, we should focus our attention on asymptotic techniques, like Judd does, albeit at the cost of assuming smaller prizes. The model in the next subsection uses the asymptotic approach pioneered by Judd and studies the ε preemption result analyzed above with a more general strategy space and with a slightly different function $g(t)$.

2.3.2. A Variation on Reinganum's Model

In this section, I make use of asymptotic theory and transform the open loop solution of Reinganum's study into a feedback form (the reader may recall that the Reinganum analysis (1981, 1982) rendered an open loop solution in which firms precommit to a path of R&D which depends only on time). The conclusion undermines the ε preemption result by Fudenberg *et al.* (1983) in the context of a richer strategy space.

I take the Reinganum analysis in which two players are engaged in a patent race to develop a new product. The winner of the race receives a patent prize equal to R and the loser gets nothing. The setting is similar to Reinganum's model. I maintain assumptions 1.2.1, 1.2.2, 1.2.6, 1.2.7, 1.4.1, 1.4.2 and I study the duopoly case.[16] The only change is in the probability of discovery, the parameter λ being a function of the accumulated level of experience. I make this transformation to find a solution in which the feedback Nash solution does not coincide with the open loop solution.

In this section, increases in the level of experience are deterministic. In contrast, in the model of the last section, improvements in position are completely random. In my view, the ideal model lies in between. Analytical solutions of the feedback Nash strategies may be very hard to calculate; indeed, they may not exist for this class of games. The way to circumvent this problem is to use asymptotic methods to analyze nearby games to the Reinganum's model. In these games, the strategies are themselves function of the state. To analyze the Markov strategies, I make suitable use of the information provided by Reinganum's mode.[17]

Let me define the probability of discovery as follows:

$$P_i(t) = 1 - e^{-\lambda(K_i(t))K_i(t)} \tag{28}$$

So that $g(t)$ becomes:

[16] Reinganum (1982) studies the multiple firm case.

[17] The model by Reinganum has a setting similar to the one stated below, the only difference being that instead of having λ a function of K_i, Reinganum assumes that λ is a constant.

$$g(t) = [\lambda'(K_i(t))K_i(t) + \lambda(K_i(t))]k_i(t)$$

where $k_i(t)$ represents the rate of knowledge acquisition. We notice that the setting for the function $g(t)$ is a hybrid between the Reinganum's model and the Fudenberg *et al.* paper. Similarly to the work of these latter authors, λ is a function of the accumulated level of knowledge. However, because in this model the strategy space is larger, the rate of knowledge acquisition also affects the conditional probability of discovery, similarly to Reinganum. This fact, together with the absence of assumption 2.1.2 precludes the ε preemption result studied above.[18]

The present value of profits for firm i for any strategy profile (f_1, f_2) becomes:

$$M^i(f_i, f_j) = \int_0^T [\text{Re}^{-[\lambda(K_i(t))K_i(t) + \lambda(K_j(t))K_j(t)]}[\lambda'(K_i(t))K_i(t) + \lambda(K_i(t))] f_i(t) - e^{-rt}e^{-[\lambda(K_i(t))K_i(t) + \lambda(K_j(t))K_j(t)]}(\frac{1}{2})f_i(t)^2]dt$$

By integrating by parts the first term of the last expression:

$$M^i(f_i, f_j) = \int_0^T [(1 - e^{\lambda(K_i(t))K_i(t)})\text{Re}^{-\lambda(K_j(t))K_j(t)} [\lambda'(K_j(t))K_j(t) + \lambda(K_j(t))]f_j(t) - e^{-rt}e^{-[\lambda(K_i(t))K_i(t) + \lambda(K_j(t))K_j(t)]}(\frac{1}{2})f_i(t)^2]dt + R(1 - e^{-\lambda(K_i(T))K_i(T)})e^{-\lambda(K_j(T))K_j(T)}$$

I look for a solution in which f_i and f_j are of the feedback form. A Nash equilibrium is then a solution in which $M^i(\hat{f}_i, \hat{f}_j) \geq M^i(f_i, \hat{f}_j)\ \forall f_i$. Let $V^i(m, K_i, K_j)$ represent the maximum value that player one can attain when both players use feedback Nash strategies and the game starts at time m with initial stock of knowledge (K_i, K_j):

[18] The value function for either of the two players is positive even in the case in which the competitor has a much higher level of capital.

$$V^i(m, K_i, K_j) = \underset{f_i}{\mathrm{Max}}\int_m^T [\mathrm{R}e^{-rt}e^{-\lambda(K_j(t))K_j(t)}(1 - e^{-[\lambda(K_i(t))K_i(t)]})$$

$$[\lambda'(K_j(t))K_j(t) + \lambda(K_j(t))]\hat{r}(t) - e^{-rt}e^{-[\lambda(K_i(t))K_i(t) + \lambda(K_j(t))K_j(t)]}(\frac{1}{2})f_i(t)^2]dt$$

$\hat{r}^i$ means that player two is choosing optimal feedback Nash strategies. For $(\hat{r}^i, \hat{r}^j)$to be a Nash equilibrium, it must satisfy the following system of Bellman equations:

$$V_t^i(t, K_i, K_j) + \underset{r^i(t, K_i, K_j)}{\mathrm{Max}} \; [V_{K_i}^i(t, K_i, k_j)r^i(t, K_i, K_j) + V_{K_j}^i(t, K_i, K_j)r^j(t, K_i, K_j)$$

$$+ R(1 - e^{-\lambda(K_i)K_i})e^{-\lambda(K_j)K_j}[\lambda'(K_j)K_j + \lambda(K_j)]r^j(t, K_i, K_j)$$

$$- e^{-rt}e^{-[\lambda(K_i)K_i + \lambda(K_j)K_j]}(\frac{1}{2})r^i(t, K_i, K_j)^2] = 0 \qquad (29)$$

with terminal conditions:

$$V^i(T, K_i(T), K_j(T)) = R(1 - e^{-\lambda(K_i(T))K_i(T)})e^{-\lambda(K_j(T))K_j(T)} \qquad (29')$$

Assuming interior solutions, the first order conditions will give the following expression:

$$r_i(t, K_i, K_j) = V_{K_i}^i(t, K_i, K_j)e^{rt}e^{[\lambda K_i(t))K_i(t) + \lambda(K_j(t))K_j(t)]}$$

By plugging the first order conditions in the Bellman equation:

$$V_t^i(t, K_i, K_j) + \frac{1}{2}(V_{K_i}^i)^2e^{rt}e^{[\lambda(K_i)K_i + \lambda(K_j)K_j]}$$

$$+ (e^{\lambda(K_i)K_i} - 1)[\lambda'(K_j)K_j + \lambda(K_j)]RV_{K_j}^j e^{rt} = 0$$

In general, closed loop solutions for the value functions that solve the above system of functional equations do not exist. I am aware of a closed loop solution only for the particular case in which the function λ is a constant. This case has already been studied by Reinganum. Let me perturb this solution a little bit and posit the following function for λ:

$$\lambda(K) = \gamma + \varepsilon K \qquad (30)$$

Then we have a one-dimensional continuum of games indexed by ε. We know the closed loop solution for the particular case in which ε is equal to zero. I can then use this result and perturb it a little bit to analyze the dynamics of nearby games for the case in which ε is greater than zero. This procedure will allow me to find strategies in a closed loop fashion that does not coincide with the open loop solution. The change will permit me to introduce real dynamic interactions in the analysis. Because $\lambda(K_i)$ and $\lambda'(K_i)$ are themselves functions of ε, the solution of the functional equation in (29) is a function of ε. Asymptotic theory uses a generalization of the implicit function theorem and Taylor series to Banach spaces.[19] As in section 2.3.1 the value function for the game when ε is greater than zero can be expressed as a Taylor series equation (see equation (27) above). For a neighborhood of 0, $(0, \varepsilon)$, the value function can be expressed in the following way:

$$V^i(t, K_i, K_j; \varepsilon) \simeq V^i(t, K_i, K_j; 0) + \varepsilon D^1(t, K_i, K_j; 0) + (\varepsilon)^2 D^2((t, K_i, K_j; 0) + \dots \qquad (31)$$

where

$$D^n(t, K_i, K_j; 0) = (\frac{1}{n!}) \frac{\partial^n V^i(t, K_i, K_j; 0)}{\partial \varepsilon^n} \qquad n = 1, 2, \dots$$

By the same token, the policy function can be approximated in the following way:

$$r^i(t, K_i, K_j; \varepsilon) \simeq r^i(t, K_i, K_j; 0) + \varepsilon G^1(t, K_i, K_j; 0) + (\varepsilon)^2 G^2(t, K_i, K_j; 0) + \dots \qquad (31')$$

with

$$G^n(t, K_i, K_j; 0) = (\frac{1}{n!}) \frac{\partial^n r^i(t, K_i, K_j; 0)}{\partial \varepsilon^n} \qquad n = 1, 2, \dots$$

[19] Existence can be proved with the use of the implicit function theorem for functional spaces. See Judd (1992).

With the aim of simplifying the calculations I will consider an autonomous version of the Reinganum's model. I take the limit when T tends to infinity and replace condition (29') with the following:

$$\lim_{T\to\infty} V^i(t, K_i(T), K_j(T)) = \lim_{T\to\infty} R(1 - e^{-\lambda(K_i(T))K_i(T)})e^{-\lambda(K_j(T))K_j(T)} = 0$$

In the autonomous case the Bellman equation is no longer a function of time:

$$rV^i(K_i, K_j) = \underset{r^i(K_i, K_j)}{\text{Max}} [V^i_{K_i}(K_i,K_j)r^i(K_i, K_j) + V^i_{K_j}(K_i, K_j)r^j(K_i, K_j)$$
$$+ R(1 - e^{-[\lambda(K_i)K_i]})e^{-\lambda(K_j)K_j}[\lambda'(K_j)K_j + \lambda(K_j)]r^j(K_i, K_j)$$
$$- e^{-rt}e^{-[\lambda(K_i)K_i + \lambda(K_j)K_j]}(\frac{1}{2})r^i(K_i, K_j)^2]dt \qquad (32)$$

When ε is equal to zero, the solution of the Bellman equation is given by the following expression:[20]

$$V^i(K_i, K_j) = (-\frac{2}{3})\text{Re}^{-\gamma(K_i + K_j)} + \text{Re}^{-\gamma K_j}$$

For notational simplicity, I will rewrite equation (23) in a general form and omit the arguments:

$$rV^i = \underset{r^i}{\text{Max}}[V^i_{K_i}r^i + V^i_{k_j}\hat{r}^j + \Pi^i] \qquad (33)$$

The reader must notice that all the terms in the last expression are function of ε. I remind the reader that at ε equal to zero, the strategies are open loop, and that $\Pi_{r^i r^j} = 0$. I use these properties in the following results. The first order conditions of the above equation will be given by the following equation:

$$V^i_{K_i} + \Pi^i_{r^i} = 0 \qquad (34)$$

To get the coefficients of the terms in ε for the asymptotic expansion, I differentiate (33) with respect to K_i and then with respect

[20] Existence can be proved with the use of the implicit function theorem for functional spaces. See Judd (1992).

to ε, then I differentiate (34) with respect to ε. I also differentiate (34) with respect to K_i and then with respect to ε. Finally, I differentiate (34) with respect to K_j and then with respect to ε. When I put together all these expressions, I get the following equation:

$$[V^i_{K_iK_i} + \Pi^i_{K_ir^i} + \Pi^i_{r^ir^i}r - \Pi_{r^iK_ir^i}r^i - \Pi_{r^iK_jr^i}r^j]\hat{r}^i_\varepsilon$$

$$+ [V^i_{K_jK_i} + \Pi^i_{K_ir^j}]\hat{r}^j_\varepsilon + \Pi^i_{K_i\varepsilon} + \Pi^i_{r^i\varepsilon}r - \Pi_{r^iK_i\varepsilon}r^i - \Pi_{r^iK_j\varepsilon}r^j = 0 \qquad (35)$$

By symmetry, the corresponding equation for player j becomes:

From (35) and (36) I can solve for $\hat{r}^i_\varepsilon$ and $\hat{r}^j_\varepsilon$. After doing the necessary calculations and plugging the functional forms for Π_s and V_s and taking the appropriate derivatives,[21] the resulting expression for the policy function $(\hat{r}^i_\varepsilon)$ is:

$$\hat{r}^i_\varepsilon = \frac{\gamma R[r((\frac{8}{9}K_i + \frac{4}{9}K_j)\gamma R + r(K_i^2 + K_j^2)\frac{2}{3})}{d}$$

$$+ \frac{\gamma^3 R^2(\frac{36}{27}K_i + \frac{24}{27}K_j) + \frac{10}{9}\gamma^2 Rr(K_i^2 + K_j^2)]}{d} \qquad (36)$$

where d is the denominator which is not a function of the state and it is positive. From the last equation we can infer the following proposition:

PROPOSITION 7. *Using asymptotic methods to analyze nearby games to the Reinganum's solution, we get the following:*

1. The leader always spends more.

2. Improvements in the state of knowledge of the rival have positive effects on the leader's spending.

3. From 1 follows that the leader has a higher probability of winning the race, although the follower does not abandon the race.[22]

[21] Remember that r^i_ε is the coefficient of the term ε in the Taylor series expansion (31'). Therefore, the Π_s and V_s are evaluated at ε equal to zero.

[22] The value function of the follower for large differences in the stock of knowledge is positive.

The last proposition illustrates the restrictiveness of the Fudenberg *et al.* (1983) paper. Similarly to them, advantages in initial conditions matter. However, I get persistency of duopoly. The second result is a consequence of the noncontractual cost property (Lee and Wilde, 1980). Also, the new specification allows us to make a more realistic study of the problem involved. We could also solve for the social planner and the collusion case and make efficiency comparisons. We may notice that the results are consistent with those of section 2.3.1 and 2.2.

There is a final remark: Suppose we introduce in the model the possibility of acquiring external technology complementary to the R&D process with the property that allows firms to increase discretely their level of knowledge at a fixed cost. Suppose also that firms exercise this option only once in the race (i.e. we are allowing for one jump in the state by paying a fixed cost). Then, due to the concavity of the value function ($V^i(K_i, K_j, \varepsilon)$) in its own state variable, the follower has a higher incentive in making those expenditures. This follows from the fact that the gains for the follower are larger than those for the leader. Therefore, if we allow for jumps in the state at fixed costs and we start the race with highly uneven levels in the state, the follower will have a higher incentive to invest in the complementary technology than the leader.

3. General Settings

So far, I have being reviewing races between firms that are not affected in their current business by the introduction of a new product. This assumption does not apply always, we can find several examples in which firms are already producing products that will be substituted by the innovation. The incentives for these firms to innovate depend on the profits that they are currently enjoying, the profits that they get if they lose the race, and the profits that they will get if they win the race.

Assume that firm i enjoys a flow of profits R^i_c, if the firm wins the race it will get a present value of profits R^i_w , and R^i_L (in present value terms) denotes the profits that it will get if it looses the race. These values may differ across firms.

In this section I will review a duopoly model with assumptions 1.2.2, 1.2.3, 1.2.4 1.2.6 and 1.2.8 of section 1.2 still valid. I use the model with flow costs (Lee and Wilde, 1980). The probability of discovery across firms is still uncorrelated. However, I allow for the possibility of imperfect patent protection. In this case the difference between loosing (R^i_L) and winning (R^i_w) will be negligible.

In this setting the expected profits for firm i are given by the following expression:[23]

$$V_m(R^i_w, R^i_L, R^i_c, c_j, r)^i = \underset{c_i}{\text{Max}}\left[\frac{\lambda(c_i)R^i_w + \lambda(c_j)R^i_L + R^i_c - c_i}{\lambda(c_i) + \lambda(c_j) + r}\right] \quad (37)$$

The respective first order conditions are given by:

$$V^i_{c_i} = \lambda(c_j)\lambda'(c_i)(R^i_w - R^i_L) + \lambda'(c_i)(rR^i_w - R^i_c)$$
$$+ c_i\lambda'(c_i) - (\lambda(c_i) + \lambda(c_j) + r) = 0 \quad (38)$$

By differentiating the first order conditions and using the implicit function theorem we can get the following expression for the slope of the optimal response function ($R^{i\prime}(c_j)$):

$$R^{i\prime}(c_j) = \frac{-\lambda'(c_j)\lambda'(R(c_j))(R^i_w - R^i_L) + \lambda'(c_j)}{\lambda(c_j)\lambda''(R(c_j))(R^i_w - R^i_L) + \lambda''(R(c_j))(rR^i_w - R^i_c) + R(c_j)\lambda''(R(c_j))}$$

By concavity of the technology, the sign of $R'(c_j)$ depends upon the sign of the following expression:

$$(\lambda'(R(c_j)(R^i_w - R^i_L) - 1)\lambda'(c_j)$$

The slope will be zero for $\bar{R}$, such that:

$$(R^i_w - R^i_L) = \frac{1}{\lambda'(\bar{R})} \quad (39)$$

[23] The reasoning to get this expression is similar to the reasoning made in section 2.2. The first term in the numerator represents expected payoffs under the different possible outcomes.

Let $R(0)$ be the solution in (38) for the case in which c_j is equal to zero, then under our assumptions $R^i(0) > 0$.

From (39) and by plugging $\bar{R}$ in the first order conditions, (38) we get claim 1:

Claim 1: If for some $0 \leq c_j < \infty$ $R(c_j) = \bar{R}$ then $R'(c_j) = 0$ and $\bar{R} = R(0)$.

PROPOSITION 8 (Beath, Katsoulacos, Ulph, 1989).

i) If $R(0) < \bar{R}$ then $\forall\ c_j \geq 0$, $R(0) \leq R(c_j) < \bar{R}$ and $R'(c_j) > 0$.

ii) If $R(0) = \bar{R}$ then $\forall\ c_j \geq 0$, $R(c_j) = \bar{R}$ and $R'(c_j) = 0$.

iii) If $R(0) > \bar{R}$ then $\forall\ c_j \geq 0$, $R(0) \geq R(c_j) > \bar{R}$ and $R'(c_j) < 0$.

The proof of the first part is simple, and the other parts follow from symmetric arguments. Suppose $R(0) < \bar{R}$ but $\exists c_j$ such that $R(c_j) = \bar{R}$, by the last claim $R(0) = \bar{R}$ which is a contradiction, hence $R(c_j) \neq \bar{R}$. At $c_j = 0$ $R(0) > 0$ and $\frac{1}{\lambda'(R(0))} < (R_w^i - R_L^i)$, because $\lambda'' < 0$ and $R(0) < \bar{R}$ consequently $R'(0) > 0$. Since $R'(0) > 0$ and $R(c_j) \neq \bar{R}$ $\forall\ c_j$ we have that $R(0) \leq R(c_j) < \bar{R}$ $\forall\ c_j$. We also notice that as c_j tends to ∞ the slope of $R(c_j)$ tends to zero. ■

Hence, under *i* in Proposition 8 the optimal response function is positively sloped tending asymptotically to 0 as c_j tends to ∞.

The comparison between $R(0)$ and $\bar{R}$ in determining the slope of the response function highlights two important effects stressed by Grossman and Shapiro (1987) (See also Katz and Shapiro, 1987), that regulate the incentives to invest: The standalone effect and the incentive to preempt. $R(0)$ represents what firm *i* would invest in R&D if the rival were investing 0. This is the so-called standalone effect, which highlights the increase in profits that the firm gets if it innovates. The term $\bar{R}$, represents the incentive to preempt, depicting the difference between winning and losing the race. We note that $\bar{R}$ represents the amount of R&D that firm *i* would do if the other firm were certain to innovate in the next instant of time. With this incentive, it does not matter what level of profits firm *i* is currently enjoying; only the difference between winning and losing matters.

If $R_w^i > R_c^i > R_L^i$, then by using the first order conditions and equation (39) it can be very easily shown that $R(0) < \bar{R}$. In this case, the incentive to preempt dominates the standalone incentive. If the rival of firm i increases its allocation to R&D, firm i responds by increasing its allocation to R&D in order to defend itself from the increased likelihood that the rival wins the race. On the other hand, if $R_w^i \simeq R_L^i > R_c^i$ then $\bar{R} < R(0)$. There is a strong externality in the R&D process. An increase in allocations to R&D by the rival induces firm i to reduce its allocation to R&D. This follows from the fact that it will get more or less the same profits regardless of who wins the race. This kind of competitive process happens whenever the loser can imitate at a low cost and very easily (i.e. that patent protection is very weak).

Beath, Katsoulacos and Ulph (1989) discuss the impact of these two effects in determining the optimal policy towards R&D in an international environment. They criticize the Brander and Spencer (1983) model that favors the subsidies of "national champions." Beath *et al.* argue that subsidies should not be implemented in the case in which the standalone incentive dominates the preemption effect. In that case, it is better not to subsidy the "national champions." The optimal policy needs the foreign firm to bear the costs of R&D and for the domestic firm to wait and imitate at low costs and without effort. A policy of subsidies would go in the opposite direction.

Bagwell and Staiger (1992) show that in a symmetric patent race with a general distribution function and lump sum cost, similar to the model in section 1.2, the optimal policy towards R&D in an international environment depends on the number of national firms involved in the race. In the case in which there is one single national firm, a policy of subsidies is optimal. Whenever there are multiple national firms, the optimal policy is to tax R&D. The results depend heavily on the negativity of the slope of the reaction function of the model in section 1.2 and on the externality in R&D that exists in market economies.[24] For a model á la Lee and Wilde (1980) (section 1.3), the results are reversed.[25]

[24] I discuss this externality more extensively in the section on efficiency below.
[25] In this case the reaction functions are positively sloped as shown in section 1.3.

3.1. *The Value of Incumbency*

An important topic of discussion in the literature on technological innovation is the analysis of the different incentives that are faced by a potential entrant into a market and a monopoly producing a product that will be substituted by the innovation. Arrow (1962) argued that for drastic innovations,[26] an incumbent monopolist would have less incentive to invest than would have a new entrant.

To study this asymmetric model, I use the setting in the last section. Let me define as firm i the monopolist and firm j the potential entrant.

As above, R_c^i denotes the flow of profits for the monopoly before innovation. R_w^i represents the present value of profits that the monopoly has if it is the first firm in producing the innovation. R_L^i is equal to the present value of profits for the monopoly if the rival innovates first. R_w^j denotes the present value of profits for the newcomer if this firm innovates first. R_c^j represents de flow of profits for the newcomer before de innovation. R_L^j is equal to the present value of profits for the newcomer if the monopolist innovates first. I assume that $R_c^j = R_L^j = 0$, i.e., the newcomer does not get any current profit and it does not get any profit if it loses the race. The expenditures made by the monopolist are given by c_i. Finally, c_j denotes the expenditures of the entrant. The expected profits for the monopolist are equal to the following expression:

$$V^i(R_w^i, R_L^i, R_c^i, c_j, r) = \underset{c_i}{\text{Max}}\left[\frac{\lambda(c_i)R_w^i + \lambda(c_j)R_L^i + R_c^i - c_i}{\lambda(c_i) + \lambda(c_j) + r}\right] \tag{40}$$

The expected profits for the potential newcomer are:

$$V^j(R_w^j, c_i, r) = \underset{c_j}{\text{Max}}\left[\frac{\lambda(c_j)R_w^j - c_j}{r + \lambda(c_j) + \lambda(c_i)}\right] \tag{41}$$

By using equation for firm j under the assumption that firm i does not produce anything:

[26] Those innovations that improve the competitive position of the innovator in such a way that it becomes a de facto monopolist.

$$\lambda'(R^j(0))(rR^j_w) + R^j(0)\lambda'(R^j(0)) - \lambda(R^j(0)) - r = 0 \tag{42}$$

we also know that for firm j, $\bar{R}$ satisfies the following equation:

$$\frac{1}{\lambda'(\bar{R})} = R^j_w \tag{43}$$

By putting together the last two equations and rearranging:

$$\left(\frac{\lambda'(R^j(0)) - \lambda'(\bar{R})}{\lambda'(\bar{R})}\right)\frac{r}{R^j(0)} + \lambda'(R^j(0)) = \frac{\lambda(R^j(0))}{R^j(0)}$$

Since $\lambda'(R^j(0)) < \frac{\lambda(R^j(0))}{R^j(0)}$, the last expression implies tha

$$\lambda'(R^j(0)) > \lambda'(\bar{R}) \Rightarrow R^j(0) < \bar{R}.$$

Therefore, the response function is positively sloped and the preemptive incentive dominates the standalone incentive for the newcomer. The first order conditions (equation 38) for the monopoly when the rival does not allocate resources to innovation are:

$$\lambda'(R^i(0))(rR^i_w - R^i_c) + R^i(0)\lambda'(R^i(0)) - \lambda(R^i(0)) - r = 0 \tag{44}$$

The equation for $\bar{R}$ satisfies the following condition:

$$\frac{1}{\lambda'(\bar{R})} = R^i_w - R^i_L$$

By putting together the last two conditions, and rearranging:

$$\left(\frac{(\lambda'(R^i(0)) - \lambda'(\bar{R})}{\lambda'(\bar{R})}\right)\frac{r}{R^i(0)} - \frac{\lambda'(R^i(0))}{R^i(0)}(rR^i_L + R^i_c) + \lambda'(R^i(0)) = \frac{\lambda(R^i(0))}{R^i(0)}$$

Since

$$\lambda'(R^i(0)) < \frac{\lambda(R^i(0))}{R^i(0)} \Rightarrow (\lambda'(R^i(0)) - \lambda'(\bar{R})) > 0 \Rightarrow R^i(0) < \bar{R},$$

the response function for firm i is positively sloped. For both the monopolist and the newcomer, the preemptive incentive dominates the standalone incentive. However, it remains to be seen which of the two firms has a net higher incentive to invest in R&D.

The first order conditions for the monopolist and the entrant can be rearranged to get the following expressions:

$$\lambda'(c_i) = \frac{\lambda(c_i) + \lambda(c_j) + r}{\lambda(c_j)(R^i_w - R^i_L) + rR^i_w - R^i_c + c_i} \tag{46}$$

$$\lambda'(c_j) = \frac{\lambda(c_i) + \lambda(c_j) + r}{\lambda(c_i)R^j_w + rR^j_w + c_j} \tag{47}$$

By concavity of the technology, the higher the benefits of becoming a leader $(R^i_w - R^i_L)$ (the preemptive effect), the greater the incentive for the monopolist to invest. By the same token, the higher the duopoly profits for the entrant, the higher the incentive to invest by the newcomer (this is also the preemptive effect of the newcomer). If in the monopoly setting there is no rent dissipation, it is reasonable to argue that $R^i_w - R^i_L > R^j_w$,[27] *caeteris paribus* the monopolist has more incentive to invest. This effect has been called the efficiency effect, Gilbert and Newberry (1982) emphasize this effect.

On the other hand, we note in equation (46) that the higher the current profits, the lower the incentive to invest by the monopoly. Since the newcomer does not enjoy current profits, it does not suffer this negative impact (see equation (47)). By using equations (42) and (44) and assuming $rR^i_w - R^i_c < rR^j_w$, we can notice that the standalone effect is higher for the newcomer (i.e. $R^j(0) > R^i(0)$). This latter result has been called the "replacement effect." Under this result, the entrant has a higher incentive to invest. If the innovation is "drastic" so that $R^i_w = 0$ and $R^i_w = R^i_w$,[28] then we can see in equations (46) and

[27] This assumption implies that a monopoly can do better than any noncooperative solution.

[28] In words, the assumption says that if the monopoly loses, it does not get any profits (its market share is completely eroded by the competitive advantage of the rival), and the return for winning is the same for both firms.

(47) that the entrant invests more, and the replacement effect dominates (Reinganu, 1983).[29] On the other hand, Fudenberg and Tirole (1986) replace the technology $\lambda(c)$ by the technology $\gamma\lambda(\frac{c}{\gamma})$, then take the limit when $\gamma \rightarrow \infty$. This modification increases the efficiency of the technology of innovation and makes the monopolist more concerned about the possibility of being preempted. The rival is going to innovate with a very high probability in the next instant of time, under the assumption that the monopolist does not dissipate rents. The efficiency effect completely dominates the replacement effect.

PROPOSITION 9: In a patent race between an incumbent and a newcomer, with assumptions 1.2.1-1.2.4, 1.2.6 and 1.2.8 from the first section plus noncontractual costs, the effort made by the monopolist will be larger than those of the newcomer if the efficiency effect dominates the replacement effect. A sufficient condition for this latter result are nondrastic innovations and very efficient technology of innovation. If innovations are drastic, so that the winner becomes a monopolist, the newcomer invest more because it does not enjoy current profits.

3.2. *Dynamic Asymmetric Models*

Although rich in their predictions and very useful, the last models were static. Harris and Vickers (1985) study a duopoly model in an alternating framework. They assume perfect information, no technical uncertainty and asymmetric incentives to do R&D. The firms may differ in their initial distances from the finishing line, their initial valuations of the prize, the efficiency of the technology of R&D and their discount rates. The four factors combine in the solution of the

[29] This can also be verified by looking at the standalone effect and the preemptive effect. From equations (43) and (45) we notice that under these assumptions the preemptive effect is the same for both firms. However, by differentiating equation (44) we notice that $\frac{dR^i(0)}{dR^i_c} < 0$, consequently the standalone effect is higher for the newcomer. Putting together the two effects, the newcomer will invest more in R&D, as Reinganum (1983) shows.

equilibrium strategies. One firm may be very close to the end, but, if the rival puts a higher prize in winning the race, the firm that was initially closer may abandon the race.

In contrast with Fudenberg *et al.* (1983), the model shows that because of the asymmetries in the initial incentives of the firms, the definition of leader is not unilaterally defined by the initial distance from the finishing line. The definition of leader depends on the concatenation of all the variables that influence the incentives to innovate.

Regarding the incumbent-newcomer discussion, they argue that "the strategic supremacy of the monopolist over potential competitors does not consist in the fact that anything they can do he can do better" (the efficiency effect p. 37), but depends on the combined result of all the incentives.

4. Efficiency Analysis

In this section I discuss the efficient allocation of resources from a social point of view and compare this allocation with the market outcome. There are several interesting questions to answer: The first concerns the importance of the structure of costs in comparing the social planner solution with the market outcome. A second question concerns the degree of riskiness that the market undertakes as compared to what is socially desirable. Finally, I address the restrictiveness of the assumption that precludes the firms from undertaking more than one project. Sah and Stiglitz (1987) have argued that whenever firms undertake several parallel independent projects aimed at the same innovation, there exists a positive probability that two or more projects will be successful,[30] and firms can perfectly price discriminate. The market outcome coincides with the social planner.

The result contrasts highly with the popular belief that argues that firms cannot internalize the effects of their decision on the whole market, and therefore, Nash equilibria in one-shot games are ineffi-

[30] They study a race in a contest setting in which the probability of winning the contest is proportional to the amount allocated to R and D. Similar models are those by Futia (1980) and Rogerson (1982).

cient.[31] The models in this section, confirm in several ways the presence of this externality. I modify slightly the Sah and Stiglitz model by analyzing an economy in which time matters together with the assumption that the first innovator gets patent rights (a stochastic patent race). The introduction of a continuous time setting precludes the possibility that two projects may be successful. If firms are allowed to undertake several projects in a game of timing, the level of effort chosen by the firms is optimal, but the number of projects is not, showing the inefficiency of the market outcome.

It is important to point out that throughout this section it is assumed that the social planner runs several parallel independent projects aimed at the same innovation, and there are no spillovers from one project to the other projects. The literature makes this assumption in order to be able to compare the socially desirable level of resources allocated to R&D with the level attained by independent firms that maximize expected profits with no spillovers. Needless to say, the lack of spillovers properly incorporated in the current literature is a serious shortcoming. Any future serious work must consider these issues. Dasgupta (1988) does this in a highly stylized model.

4.1. *Contractual Costs*

Assumption 4.1.1: There is free entry to the industry and firms will come into the market until the level of expected profits are driven to zero.

The aim of the last assumption is to have a benchmark to compare the number of projects (in this setting equal to the number of firms) with the social planner outcome. The idea behind this assumption is that the industry is in stable equilibrium only when expected profits are equal to zero. This assumption may be sensible for static analysis.

Assumption 4.1.2: Suppose that $\lambda'(c) > 0$ and $\lambda''(c)$ greater or less than 0, according to c less or greater than $\tilde{c}$. In other words, there are initial increasing returns and then decreasing returns.

The last assumption is a variation on assumption 1.2.8 in section 1.2. This new postulate is crucial to highlight the differences between

[31] See Tirole (1988).

the market and the social planner outcome for the contractual cost case. Under global decreasing returns to scale, only when n (the number of firms) tends to infinity can expected profits be driven to zero, this leaves us with no room for comparison in regard with the number of projects. I maintain assumptions 1.2.1 to 1.2.7 in section one. For the noncontractual cost case, the inclusion of a fixed cost will give a finite number of projects for the social planner solution and for the market solution. However, I maintain assumption 4.1.2 also for the noncontractual cost case, to be able to compare with the lump sum model. The reader may notice that a technology of innovation with global increasing returns will commit the social planner to undertake just one project. This happens for both the contractual cost case and the noncontractual cost case.

In order to compare the social solution with the market outcome, we must define social benefit. There is no exact measure of the benefits of innovation because it is difficult to measure the demand for a new product. If the new product substitutes an existing one, the social benefit may be lower than the private benefits. Alternatively, due to the incapability of the private producer to extract the whole consumer surplus, the social benefit may exceed private benefits. Let us assume that there is no substitution of current products and that the innovator can behave as a perfect discriminating monopolist. In this case, social benefits (s) will equal private benefits (R, $S = R$). Since society does not care which firm succeeds, the social planner is only concerned with $n\lambda(c)$, the aggregate probability of innovation. Expected social benefit is:

$$W = \frac{n\lambda(c)S}{n\lambda(c) + r} - nc$$

The first order conditions for n^w and c^w are:

$$c^w = \frac{Sr\lambda(c^w)}{(n^w\lambda(c^w) + r)^2}\,\frac{1}{\lambda'(c^w)} = \frac{Sr}{(n^w\lambda(c^w) + r)^2} \tag{48}$$

From the last equations we can obtain very easily that $c^w = \bar{c}$ (where $\bar{c}$ is given by the expresion $\frac{\lambda(\bar{c})}{\bar{c}} = \lambda'(\bar{c})$). The level of effort that maximizes social welfare is such that it minimizes social average costs.

This result is valid regardless of the level of s. If the level of s changes, the social planner will respond by changing the number of projects.

Now consider the market outcome. In this comparison, assumption 4.1.1 is instrumental. Under these circumstances, by setting equation (3') (the profit equation in section 1.2) equal to zero, and by using equation (4) (the first order conditions for the market), we get the following result:

$$\frac{\lambda(c_i^{mar})}{c_i^{mar}} < \lambda'(c_i^{mar})$$

According to the last equation, c^{mar} is smaller than c^{w}. This result is helpful in seeing whether the market yields as an outcome a higher number of projects than the social planner. If we compare equation (48) with equation (4) and we use the fact that $c^{w} > c^{mar}$, we will se that $n^{w} < n^{mar}$. The intuition follows from the fact that firms do not care about the duplicity of efforts. Therefore, they will spend more on R&D than the level chosen by the social planner for the market structure given by n^{w}[32] (the number of firms chosen by the social planner). We also know that $c^{mar} < c^{w}$, by proposition one, the optimal response function is negatively sloped in terms of the degree of rivalry (the number of firms). This can only happen if the number of projects is larger in a competitive environment than in the social planner case. Proposition 10 summarizes these findings:

PROPOSITION 10 (Loury, 1979). *If assumptions 1.2.1-1.2.7 and assumptions 4.1.1-4.1.2 hold, the market outcome will give a larger number of firms than the social planner, with each firm working at a lower level of effort than the socially desirable level.*

As Kamien and Schwartz have pointed out, "This result is reminiscent of Chamberlin's conclusion... free entry results in too many suboptimal plants" (Kamien and Schwartz, 1982).

[32] i.e. $c^{mar}(n^{w}) > c_{w}(n^{w})$.

4.2. *Noncontractual Costs*

In this case the social planner will maximize:

$$W^* = n[\frac{S\lambda(c) - c}{n\lambda(c) + r} - F] \tag{51}$$

By calculating the first order conditions, and plugging these in the last equation:

$$W^* = n[\frac{\lambda - \lambda'(c)c}{r\lambda'(c)} - F]$$

For W^* to be positive, we need $\frac{\lambda(c)}{c} > \lambda'(c)$, so that in the case of noncontractual costs is bigger c^w than $\bar{c}$.[33]

It can be proved that $n^w < n^{mar}$ and $c^{mar}(n^{mar}) > c^w(n^w)$[34] (where n^{mar} is given by assumption 4.1.1). The following proposition summarizes this argumentation:

PROPOSITION 11 (Lee and Wilde, 1980). *With contractual costs and assumptions 1.2.1-1.2.4, 1.2.6-1.2.7 in the first section and assumption 4.1.1-4.1.2, the social planner will set* $c^w(n^w) < c^m(n^m)$, $c^w > \bar{c}$ *and* $n^w < n^m$.

Both the lump sum model and the noncontractual model yield a larger number of projects than the socially desirable level.[35] However, the level of effort differs between the two models. The lump sum cost

[33] The result is due to the fact that $F > 0$. When $F = 0$ the result is identical to the contractual cost case.

[34] The interested reader should see Lee and Wilde (1980). The proof is inessential and long.

[35] Battacharya and Mookherjee (1986), and Dasgupta and Maskin (1987) have studied models in which there is a whole portfolio of projects of R&D available for each firm, with all projects aimed at the same innovation. With this setting, they study the issue of correlation. In their model, there exists an externality in the sense that firms choose too much correlation in comparison with the socially desirable level. Firms will choose similar projects to their rivals' with the aim of reducing the probability of losing the patent. If a firm chooses a project uncorrelated to its rivals', a failure by the firm will most likely be accompanied by a success of the rivals. From the social point of view, the projects chosen should be evenly separated in the space of projects. As in the main text, the externality arises because firms care only about their own success.

model yields a lower effort for the market than the social planner level. In contrast, the noncontractual cost model yields a higher level of effort for the market. This latter result extends to more dynamic settings such as those studied by Reinganum (1981) and Judd (1986). Because of the dynamic structure of those models, their structure of costs is obviously noncontractual.

4.3. *Dynamic Settings and Varying Degrees of Riskiness*

The latter models were set in a static framework, because costs are still noncontractual in a dynamic setting, the intuition tells us that Proposition 11 should still be valid. Indeed, this result is still valid in Judd's model. His work compares the social planner outcome with the market outcome and distinguishes between two types of projects that vary in riskiness.[36] The model allows us to see whether resources allocated to risky projects by the market are excessive when compared with the socially desirable level. Following the procedure outlined in section 2.3.1 we can find in an analytical way the value function and the policy functions of the social planner and compare this outcome with the market outcome. The comparison with the market outcome will yield the following proposition:

PROPOSITION 12 (Judd 1986). *In a dynamic setting with two firms and assumptions 1.2.1, 1.2.2 and 1.2.7, quadratic costs in R&D, and the postulate that R is small, the following results hold:*

1. The market yields excessive effort in all projects compared with the social planner.

2. The market gives excessive allocation to risky projects.

The first conclusion emerges because of the noncontractual property of the cost function in dynamic settings. The second result is similar to Klette and De Meza (1986), Dasgupta and Maskin (1987)

[36] There is no equivalent of assumption 4.1.1 in dynamic settings. In dynamic settings it is difficult to have an analytical solution of the value function. This fact makes it more difficult to endogeneize the number of firms. In dynamic settings efficiency analysis is usually done for a given market structure.

and Bhattacharya and Mookherjee (1986). It represents an externality in the sense that the individual firm does not take into account the benefits of the rival in allocating its resources to the risky project. This fact makes the firm spend more on risky projects than the socially desirable level.

4.4. *Multiple Projects and Multiple Possible Innovators*

Proposition 10 has been challenged by Sah and Stiglitz (1987) in a slightly different setting. They argue that this model restricted the choice of the firms to one project without justification. If we do not do so, the market will yield the optimal number of projects and the level of expenditure per project will be set at the socially efficient level. Further, this result will not be affected by the number of firms. This striking result criticizes the widely held view (in static settings) that there is an optimal number of firms such that social welfare is maximized.[37] Under the Sah and Stiglitz model, the important variable is the total number of projects undertaken by the market. When firms are allowed to undertake several projects, Sah and Stiglitz argue that the decision that one firm makes in undertaking the marginal project depends upon market parameters, the size of the rewards, the technology and the number of projects undertaken by the market. If the marginal project yields positive expected marginal benefits for the firm (taking into account all the other projects that the market undertakes), the firm will invest in that project. It does not matter whether many of firms or just a few firms are undertaking the other projects. What matters is the number of projects undertaken by the market, not how these projects are allocated between firms.

To make a good assessment of Sah and Stiglitz challenge, let us consider briefly their model and assume the following postulates:

4.3.1 The winner takes all the benefits. Sah and Stiglitz argue that Bertrand competition is a sufficient condition for the latter to be true.

[37] See Loury (1979), Lee and Wilde (1980), Dasgupta and Stiglitz (1980) and Kamien and Schwartz (1982).

4.3.2 Firms can either win the race, lose the race or tie in the race (there can be more than one innovator). The winner is decided like a lottery in which the probability of success for a project depends upon the resources allocated to that project. In case of more than one firm innovating (there is no time in the economy), Bertrand competition will take away all the benefits.

4.3.3 All firms are equal in terms of technology (symmetry).

4.3.4 There is an independent probability of success for each project (no spillovers across projects).

4.3.5 All projects are aimed at the same innovation.

4.3.6 Suppose that the probability of innovation $p(\cdot)$ has the following property: $p'(c) > 0$ and $p''(c)$ greater or less than 0, according to c greater or less than $\bar{c}$. In other words, we have initial increasing returns and then decreasing returns.[38]

Define the following variables:

$$a_i = \prod_{j \neq 1} \prod_{s=1}^{k_i} (1\ p(c_{js}))$$

$$\lambda_i = 1 - \prod_{s=1}^{k_i} (1 - p(c_{is}))$$

c_{is} represents the amount allocated by firm i to project s. a_i represents the probability that the rivals of i fail, λ_i is the probability that firm i succeeds and k_i represents the number of projects undertaken by firm i. Expected benefits are given by the following expression:

$$V^* = a_i \lambda_i R - k_i(c_i)$$

Where R represents the size of the rewards. Taking the first order conditions, imposing the symmetry conditions and simplifying:

[38] This assumption is not essential for the validity of the Sah and Stiglitz result. Because of the independence of the projects undertaken by a single firm, the aggregate probability of success does not have the constant returns to scale property in the number of projects. See Footnote 2 in Sah and Stiglitz. I make this assumption to make easier the comparison with the model in the next subsection.

$$R(1-p(c))^{Nk-1}p_c = 1$$

$$R(1-p(c))^{Nk-1}p = c$$

where N is the number of firms. From the last equations we get easily that $p_c = \frac{(c)}{c}$, i.e. the level of resources allocated to each project is efficient, in contrast with the results of section 4.1 and 4.2.

Alternatively, let us analyze the social planner problem. As before, s represents social benefits. g represents the aggregate probability of success. Social benefits may be written as follows:

$$W = Sg - Nkc$$

where g is equal to $1-(1-p(c))^{NK}$, the first order conditions with respect to k and c:

$$S(1-p(c))^{Nk-1}p_c = 1$$

$$S(1-p(c))^{Nk-1}p = c$$

If $S = R$, then the social outcome and the market outcome are the same. The following proposition summarizes these findings:

PROPOSITION 13 (Sah and Stiglitz, 1987). *With assumptions 4.3.1 to 4.3.5, especially the possibility for two or more projects to be successful, the following properties hold:*

1. *The level of effort for each firm is set at a level such that* $P_c = \frac{P(c)}{c}$.
2. *The social planner outcome corresponds to the market outcome if* $R = S$.

The critical assumption that drives the second result is the statement that competition takes place in a contest setting in which timing does not matter and more than one project can be successful. If we consider stochastic patent races, with each firm trying to finish in first position and get the patent rights (winner takes all), the Sah and Stiglitz proposition no longer holds. In this setting, firms will choose an efficient level of expenditure, but the number of projects will exceed the social optimum. I discuss this issue in the next section.

4.5. *Multiple Projects, the Role of Timing*

Consider the model in section 1.2 (lump sum costs) and suppose that all the assumptions in that section are valid. Let us assume further, as in the Sah and Stiglitz model, that firms are allowed to undertake several projects. We may notice immediately that the only important assumption that I am changing is that the race is going through time. The assumption of lump sum costs helps to make the comparison equivalent to the Sah and Stiglitz model, because it makes the cost expenditures independent of the duration of the race. We know this is trivially true for an economy without time. On the other hand, the assumption that the winner takes all is better justified in this case by assuming that a patent right gives the winner the privilege of exploitation for a certain amount of time. I also maintain assumption 4.2.1. Under the last setting, expected payoffs are given by the following expression:

$$V^i = \frac{Rk^i\lambda(c)}{\mu + k^i\lambda(c) + r} - k^i c$$

where k^i represents the number of projects undertaken by firm i,

$$\mu = \sum_{j=1\, j\neq i}^{N} k^j\lambda(c_j)$$

denotes the other firms' number of projects times their probability of success. Castañeda (1993) proves the following proposition:

PROPOSITION 14 (Castañeda, 1993). *If a patent race has the following properties: I. Firms are allowed to take several projects in the market. II. Firms have lump sum costs. III. Assumptions 1.2.1-1.2.7 and 4.1.2 hold, then the following implications hold: The level of effort chosen by the firm is optimal, but the number of projects undertaken by the market exceeds the socially optimal number.*

In a world with no time, the probability that two or more projects are successful is strictly positive. A firm undertaking a marginal project will have a positive benefit only if all the other projects undertaken in the market (including those undertaken by the firm) fail. The fact that two or more projects already undertaken by the firm may be successful undermines the firm's incentives to undertake the marginal project. In contrast, in the model of this section in which firms compete through time for the right to become a monopoly (a patent), a higher number of projects increases the likelihood of being first and obtaining the patent rights. Because each firm wants to be first, they undertake a number of projects that exceeds the socially desirable number. Firms do take into account in their marginal decisions the number of projects undertaken by other firms whenever we have a noncooperative solution. But the market outcome finishes up with more projects than the socially optimal level.

It is interesting to contrast the results of this proposition with those of Sah and Stiglitz. I believe that it is more relevant to model R&D processes in a game of timing with the winner acquiring the right to become a monopoly. In this sense, the Sah and Stiglitz paper is less relevant. Nonetheless, the important result of their paper is that regardless of the market structure, the market will always yield the same number of projects (the invariance theorem). In the context of this model this holds only whenever we have a noncooperative Nash solution, Castañeda (1993) shows that the invariance theorem does not hold when we pass from one firm to two or more firms, although it is valid every time we compare two market structures in a noncooperative solution.

If the timing of innovation matters, the objective of policy is to grant patent rights for a duration of time that can reduce expected profits in such a way as to reduce the number of projects that firms undertake.[39]

Finally, it is interesting to contrast the invariance theorem with the previous belief about the relation between market structure and innovation. The previous belief considered the number of firms as an important determinant of the incentives to innovate and the expectations of monopoly rents were considered as crucial for undertaking R&D. The result of this model undermines the structure of the market in a noncooperative environment as an important variable for

[39] Remember that we are assuming that social benefits are equal to private benefits.

determining the level of R&D, but the possibility of getting monopoly rents is still important.

5. Conclusions

In this paper I have studied the main theoretical contributions that study how firms invest in R&D in a strategic setting, with time as an important factor. The paper shows how the original contributions have been modified in several ways to explain in more general settings how the innovation process works: First, because competition for innovation is basically a dynamic process in which firms react to changes in the competitive process as the race unfolds, the literature has dynamized the seminal models by studying models in which firms condition their behavior on their achievements and those of their rivals. Second, the literature has studied models in which firms have different incentives to innovate (i.e. asymmetric models). I also studied the efficiency properties of the market, showing that the market outcome allocates excessive resources to R&D.

In the first section, I analyzed the seminal ideas of the game theoretic approach to R&D processes (Loury, 1979), Lee and Wilde (1980) and Reinganum (1981, 1982). The difference in the specification of the cost function yields two contrasting results in terms of the response of a representative firm to a change in the degree of rivalry (Propositions 1 and 2). When the R&D process is such that lump costs are the main determinant of the conditional probability of discovery (contractual costs), an increase in the degree of rivalry diminishes the effort realized by a representative firm. When the conditional probability is dependent on a flow cost which is paid only until the race is over, an increase in rivalry stimulates the amount allocated to R&D by a representative firm. These contrasting results have implications for the efficiency properties of the models and for the results of multistage games that entail these one-stage games as constituents of the more general dynamic games.

The first attempt to introduce dynamics was the differential game approach initiated by Reinganum (1981, 1982). In this approach, the probability of discovery increases with the level of effort accumulated by the firm (equation (10)). However, due to the memorylessness property of the exponential distribution function, the model yielded an open loop solution. The only achievement of this approach is to obtain a time varying policy function. The attempts to introduce dynamics and characterize the game with feedback strategies fail. In few words the model "yielded a static solution" (Fudenberg and Tirole, 1986).

To construct dynamic models, the literature has pursued several strategies: 1) To include stages in the race. 2) To posit the conditional probability of discovery $g(t)$ as dependent upon the level of effort accumulated. 3) To include in g_t, besides the level of experience, the effort on R&D chosen by the firm at time t.

If the conditional probability of discovery depends just on the level of experience accumulated and the strategies are restricted to a constant level for the whole game, we get the monopoly outcome (ε preemption). An advantage on the initial level of knowledge by any firm, no matter how small it is, will preclude the rival from doing R&D right at the beginning. However, Lippman and McCardle (1988) show that this result depends crucially on the implicit assumption that allows firms to make a choice at each moment in continuous time. If we restrict the analysis to models in which the decision period lasts for an amount of time , the result no longer holds.

On the other hand, if $g(t)$ depends on the level of effort and the allocations to R&D made by firms, and if we use asymptotic techniques, the ε preemption result vanishes and we get the persistence of competition even with large differences in initial conditions. Furthermore, if we introduce complementary technology that allows firms to make one jump in their state of knowledge at a fixed cost, large initial conditions differences will be ameliorated by the investment of the follower in the complementary technology. The results depend heavily on small effects of the accumulated level of knowledge on the function $g(t)$.

One of the simplest ways to introduce dynamics in patent races is to study a two-stage version of the noncontractual cost model (Grossman and Shapiro, 1987). Under the noncontractual cost assumption, the model shows that the leader always spends more,[40] that

[40] This result was also obtained by Judd (1991) and in section 2.3.2 of this paper.

the intensity of rivalry is higher in the second stage and that if the race becomes tied, both the leader and the follower speed up. If costs are contractual, the results change, because the slope of the response function changes.

If we want to study more complex patent races we must resort to perturbation methods (Judd, 1986). With these techniques we can analyze multistage models with several projects that differ in risk and the strategy spaces do not have to be restrictive. However, we need to assume a small prize for the technique to work . The model allows us to see how the allocation to risky and nonrisky projects varies when the conditions of the race change. It also shows the excessive allocation to risky projects by the market, in comparison with the socially desirable level. We have a better perspective of the usefulness of perturbation methods when we see that multistage versions of the noncontractual cost models (Harris and Vickers, 1987) that do not use these approaches do not yield more qualitative results than those already obtained by the two-stage version (Grossman and Shapiro, 1987).

In section 3 I studied asymmetric models. The interaction of the "standalone" effect and the "preemption" effect affects the slope of the response function and the optimal policy of taxation in international contexts. I also applied this terminology to the study of the different incentives that a monopoly and a newcomer have to innovate. We noticed that the monopolist may have more incentive to invest in R&D because it internalizes all the appropriate decisions for the market ("the efficiency effect"). On the other hand, the larger the current profits, the smaller the incentive of the monopolist to invest ("the replacement effect"). If innovations are drastic, the replacement effect dominates the efficiency effect. If the technology of innovation is extremely efficient, the efficiency effect dominates.

In the analysis of efficiency of the market outcome in the noncontractual and contractual cost models, we noticed that the market yields an excessive number of projects (firms). However, the lump sum model generates a lower level of effort for the market in comparison to the social planner solution. In contrast, the flow cost model yields a higher level. The results hold assuming that social benefits are equal to private benefits. When firms invest in multiple projects and consider the possibility of multiple projects to be successful (Sah and Stiglitz, 1987), the only possibility for the market outcome to differ from the socially optimal level is for private benefits to differ from social benefits. In a framework in which firms can undertake multiple projects with only one possible innovator (a stochastic patent race), the market outcome undertakes an efficient level of effort, but the number of projects is excessive even in the case in which private benefits are equal to social benefits.

The survey shows the presence of a negative externality in the market outcome in comparison with the social planner solution. This externality is present as well in one shot Cournot and Bertrand models. The externality arises from the fact that in making its decision, the firm does not take into account the impact of its decision on the whole market only considering the impact on its profits. As a result, we may notice excessive resources allocated to innovation in comparison to what should be socially desirable. If firms cannot perfectly price discriminate, they may have less incentive to innovate than the social planner. The role of policy is to grant patent rights for a duration of time that accounts properly for the two effects.

In the agenda for future research figures prominently the need to incorporate spillovers in the context of models with multistage races and truly feedback solutions.[41] There is consistent evidence that "imitative research is a pervasive phenomenon" (Dasgupta, 1988, p. 74). One possibility is to make the function $g(t)$ in section 2.3 not only a function of the level accumulated by firm i but also in some way of the level accumulated by j.[42] We should also allow for correlation among projects undertaken inside the firm. This task appears analytically difficult. However, I believe that the use of asymptotic techniques in a framework similar to Judd (1986) may be fruitful.

[41] Reinganum (1982) incorporates spillovers in an open loop model.

[42] It is important to point out that spillovers in static models can be easily captured in the profit function. However, when we go into a dynamic setting it is more reasonable to incorporate them in the technology of innovation to see how they affect the dynamics of the game.

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