AN ECONOMIST'S GUIDE TO THE KALMAN FILTER

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Resumen: Casi desde su aparición, el Filtro de Kalman (KF) ha sido utilizado con éxito en ingeniería de control. Desafortunadamente, muchos de sus principales resultados han sido publicados en revistas de ingeniería, con lenguaje, notación y estilo propios de tal disciplina. En este trabajo, queremos presentar el KF en forma atractiva para los economistas utilizando teoría de la información e inferencia bayesiana.

Abstract: Almost since its appearance, the Kalman Filter (KF) has been successfully used in control engineering. Unfortunately, most of its important results have been published in engineering journals with language, notation and style proper of engineers. In this paper, we want to present the KF in an attractive way to economists by using information theory and Bayesian inference.

1. Introduction

The Kalman Filter (KF) introduced by Kalman (1960), Kalman and Bucy (1961), and independently by Swerling (1959) and Stratonovich (1960),

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has been successfully used in control engineering. Many of its numerous applications in engineering can be appreciated in a special volume edited by Sorenson (1985).

In spite of the interest in control theory by economists, the importance of the KF in modeling economic systems has not been fully appreciated. Applications of the KF in economics are rather scarce: Athans (1974), Burmeister and Wall (1982), Burmeister, Wall and Hamilton (1986), Sargent (1989), and Basar and Salmon (1989), among others.

Some of the potential uses of the KF can be illustrated in the following instances: Most of the econometric models assume fixed coefficients becoming, in many cases, susceptible to Lucas' critique on econometric policy evaluation (1976). In this respect, the KF provides a methodology to deal with the estimation of time varying coefficients in regression models.

In most of the economic models stated under a stochastic optimal control framework, the state variables are supposed to be observable. However, there are many situations in which the state variables cannot be directly observed, but only through indirect measurements. The main feature of the KF is that it still allows us to analyze dynamic economic systems with indirect measurements of the unobserved state variables.

Our approach in presenting the KF uses information theory which has been extensively applied in economics; we mention, for instance, Theil, Scholes and Uribe (1965), Theil and Uribe (1965), Uribe, De Leeuw and Theil (1965), Cozzolino and Zahner (1973), Akaike (1981), Kapur (1990), and Venegas and de Alba (1992).

We start off the recursive procedure of the KF by determining, via information theory, an estimator of the initial distribution when there is information in terms of moments (Venegas, 1992, 1990 and 1990a), and then we use Bayesian inference to state the updating process of the KF which is based on the same principle as that of the sequential learning mechanism used by Lucas (1973) in obtaining his model of the Phillips' curve. We think that under this framework the presentation is more attractive to economists. It is worth pointing out that our approach to obtain the KF is simpler than those from Ho and Lee (1964), and Meinhold and Singpurwalla (1983).

We emphasize the potential use of the KF in modeling economic systems. We provide a Bayesian sequential method based on the KF to test convergence of rational expectations in the absence of sufficient side conditions. Our proposal differs from that in Burmeister and Wall (1982) in two main respects: First, we introduce entropy maximizing behavior. Secondly, we state a sequential test, which may save computations when rejection occurs at early stages.

We also discuss the KF relationship with the class of stochastic optimal control problems with a quadratic index of performance and subject to constraints in terms of state-space equations. Using this framework, we study a central planner problem under uncertain inflation.

The paper is organized as follows: In section 2, we introduce the measurement and state equations. Through section 3 we list a number of models in the econometric literature that can be written in terms of the measurement and state equations. In section 4, we present the KF methodology. Here, we briefly outline the maximum entropy principle and the Bayesian approach to statistical inference. The former will provide an estimator of the initial distribution to start off the sequential procedure of the KF, and the latter will update information recursively. In section 5, we study the connection of the KF with the generalized least squares methodology. Through section 6, we provide a method based on the KF to test convergence of rational expectations. In section 7, we discuss the relationship of the KF with stochastic optimal control given by the separation theorem. In section 8, we study a central planner problem in an economy without capital resources and uncertain inflation. In the last section, we state a set of conclusions and delimitations of our work.

2. The State-space Representation

Let $Y_1, Y_2, \ldots, Y_t$ be a set of indirect measurements, from a polling system or a sample survey of an unobserved state variable β_t. The objective is to make inferences about β_t. We may think of Y_t and β_t as either scalars or vectors with dimensions which may be the same or different. However, in this section we focus our attention in the multivariate case, and in the next section we give examples of the univariate case. All vectors and matrices below are assumed to be of consistent dimensions. The relationship between Y_t and β_t is specified by the measurement equation, sometimes also called the observation equation:

$$Y_t = X_t\beta_t + \varepsilon_t, \tag{2.1}$$

where X_t is a matrix of known parameters, and the ε_t is the observation error distributed as $N(0, \Sigma_{\varepsilon_t})$, with Σ_{ε_t} known. Since the variance changes over time we have, in general, a heteroscedastic error model. Notice that the main difference between the measurement equation and the conventional linear model is that in the former, the coefficient β_t changes with time.

The most popular dynamic extension of the error term in the conventional linear model states that

$$\begin{cases} Y_t = X_t\beta + \varepsilon_t, \\ \varepsilon_t = Z\varepsilon_{t-1} + \eta_t, \end{cases} \tag{2.2}$$

where Z is a matrix of known parameters, and η_t is distributed as $N(0, \Sigma_\eta)$. Notice that β and Σ_η are time invariant. The KF will not be concerned with the dynamics of the error term, ε_t, as in (2.2), but instead with the dynamics of the state variable, β_t, in (2.1), this being the other essential difference from the conventional linear model. We suppose that the dynamic behavior of the state variable β_t is driven by a first order autoregressive process, that is,

$$\beta_t = \mu_{t-1} + Z_t\beta_{t-1} + \eta_{t-1},$$

where the drift μ_{t-1} is a vector of exogenous or predetermined variables, Z_t is a matrix of known parameters and $\eta_t \sim N(0, \Sigma_{\eta_t})$ with Σ_{η_t} known. Or even more generally,

$$\beta_t = \mu_{t-1} + Z_t\beta_{t-1} + L_t u_{t-1} + \eta_{t-1}, \tag{2.3}$$

where L_t is a known matrix that relates the control inputs, u_{t-1}, to β_t. Equations (2.1) and (2.3) are known in the literature as the state-space representation of the dynamics of β_t. Throughout the paper, we shall assume that β_0, ε_t and η_t are independent random vectors, and for the time being $L_t = 0$; further consideration of the control variable will be made in section 7.

We might state nonlinear versions of (2.1) and (2.3), but this would not make any essential differences in the analysis.

3. Some Econometric Models that Accept a State-space Representation

In this section we list some of the models in the econometric literature that can be written in terms of the measurement and state equations. Some of them contain univariate error terms. Their multivariate extensions are straightforward.

The first model we mention is the random coefficient model from Hildreth and Houck (1968), which consists of the following two equations:

$$\begin{cases} Y_t = X_t \beta_t, \\ \beta_t = \bar{\beta} + \eta_t, \end{cases}$$

where the η_t's are independent and identically distributed random variables $N(0, \sigma_\eta^2)$ (cf. Swamy and Mehta, 1975). In this case, $\bar{\beta}$ is called the nonstochastic mean response coefficient.

The model studied in Harvey and Phillips (1982) is an extension of the previous one with the same measurement equation, but the state equation is instead

$$\beta_t - \bar{\beta} = \rho(\beta_{t-1} - \bar{\beta}) + \eta_t,$$

where $|\rho| < 1$, and the η_t's are independent and identically distributed random variables $N(0, \sigma_\eta^2)$. Note that when $p = 0$ the model reduces to the Hildreth-Houck model.

We also mention the Cooley and Prescott's (1973) model

$$\begin{cases} Y_t = X_t \beta_t, \\ \beta_t = \beta_t^p + \eta_t, \\ \beta_t^p = \beta_{t-1}^p + \nu_t, \end{cases}$$

where $E\{\eta_t \nu_s\} = 0$ for all t, s. The quantity β_t^p is called the permanent component of the parameter. Here, the parameter variation is of two types, permanent and transitory, the former allowing some persistent drift in the parameter values.

Finally, we point out that the ARMA(p, q) processes

$$(1 - \varphi_1 L - \varphi_2 L^2 - \ldots - \varphi_r L^r)Y_t = (1 + \theta_1 L + \theta_2 L^2 + \ldots + \theta_{r-1} L^{r-1})e_t$$

where $r = \max\{p, q+1\}$, $\varphi_i = \theta$ for $i > p$, $\theta_j = \theta$ for $j > q$ and the e_t's are independent Gaussian variables with mean zero and variance σ^2, can be also rewritten in terms of measurement and state equations by setting $X_t = [1, \theta_1, \ldots, \theta_{r-1}]^T$, $\beta_t = [\beta_{1t}, \beta_{2t}, \ldots, \beta_{rt}]^T$ (the superindex T denotes the usual vector or matrix transposing operation),

$$Z_t = \begin{pmatrix} \Phi_{r-1} & \varphi_r \\ \mathbf{I} & \mathbf{O} \end{pmatrix} \text{ and } \eta_{t-1} = \begin{pmatrix} e_t \\ \mathbf{O} \end{pmatrix},$$

where $\Phi_{r-1} = [\varphi_1, \varphi_2, \ldots, \varphi_{r-1}]$, $\mathbf{I}$ is the identity matrix of order $r-1$ and $\mathbf{O}$ is a column vector of $r-1$ zeros (cf. Hamilton, 1994).

4. Kalman Filtering

In order to present the KF in a simple way, we first outline the maximum entropy principle and the Bayesian approach to statistical inference. The former will provide an estimator for the initial prior distribution to start off the sequential procedure of the KF, and the latter will provide the recursive updating of information of the KF.

The principle of maximum entropy (Jaynes, 1957) provides a general method of inference about an unknown density, $p(\beta_0)$ when there is information about $p(\beta_0)$ in terms of moments. The principle states that among all compatible distributions with the available information, we should choose as estimate, $\pi(\beta_0)$, for $p(\beta_0)$, the one with the greatest entropy $\int_{-\infty}^{\infty} \log(\pi(\beta_0))\pi(\beta_0)d\beta_0$.

Suppose that at time $t = \theta$ the available information is given by $\hat{\beta}_0$ and $\hat{\Sigma}_0$, the mean and variance of β_0, respectively. We may then use the principle of maximum entropy to find an estimate, $\pi(\beta_0)$ of the prior distribution of β_0 that takes into account the given information by solving the problem:

$$\text{Maximize} \int_{-\infty}^{\infty} \log(\pi(\beta_0))\pi(\beta_0)d\beta_0,$$

$$\text{subject to: } \begin{cases} \int_{-\infty}^{\infty} \pi(\beta_0) d\beta_0 = 1, \\ \int_{-\infty}^{\infty} \beta_0 \pi(\beta_0) d\beta_0 = \hat{\beta}_0, \\ \int_{-\infty}^{\infty} (\beta_0 - \hat{\beta}_0)(\beta_0 - \hat{\beta}_0)^T \pi(\beta_0) d\beta_0 = \hat{\Sigma}_0. \end{cases}$$

The first order condition of the above calculus of variations problem of maximum entropy is given by

$$\pi(\beta_0) \propto \exp\{\lambda + \Lambda^T \beta_0 + (\beta_0 - \hat{\beta}_0)^T L (\beta_0 - \hat{\beta}_0)\}, \tag{4.1}$$

where λ is a scalar, Λ is a vector and L is a symmetric matrix. By substituting (4.1) in the constraints, we can show that $\beta_0 \sim N(\hat{\beta}_0, \hat{\Sigma}_0)$ (see Venegas, 199θ).

Suppose now that, at time t, we wish to make inferences about the conditional state variable $\theta_t = \beta_t \mid I_t$, where $I_t = \{Y_1, Y_2, \ldots, Y_{t-1}\}$. The Bayesian approach is to assume that there exists a prior density $\pi(\theta_t)$ describing initial information. Once a prior has been prescribed, the information provided by the measurement Y_t, with density $p(Y_t \mid \theta_t)$, is used to modify the initial knowledge, as expressed by $\pi(\theta_t)$, via Bayes' theorem to obtain a posterior distribution of θ_t, namely

$$p(\theta_t \mid Y_t) \propto p(Y_t \mid \theta_t) \pi(\theta_t). \tag{4.2}$$

The normalized posterior distribution is then used to make inferences about θ_t.

We are now in a position to state the recursive updating procedure of the KF. In the rest of the paper, the drift μ_t, introduced in section 2, will be left out since it makes no difference in the succeeding analysis.

At time $t = 0$, the maximum entropy estimator for the initial distribution, $N(\hat{\beta}_0, \hat{\Sigma}_0)$, describes the initial knowledge of the system. Proceeding inductively, at time t, $\hat{\beta}_{t-1}$ and $\hat{\Sigma}_{t-1}$ become available information and therefore prior knowledge at time t is represented by

$$\theta_t = \beta_t \mid I_t \sim N(Z_t \hat{\beta}_{t-1}, M_t), \tag{4.3}$$

where

$$M_t = Z_t \hat{\Sigma}_{t-1} Z_t^T + \Sigma_{\eta_{t-1}}. \tag{4.4}$$

The sampling model (or likelihood function) is determined by

$$Y_t \mid \theta_t \sim N(X_t \beta_t, \Sigma_{\varepsilon_t}). \tag{4.5}$$

The posterior distribution is then obtained by substituting both (4.3) and (4.5) in (4.2), so

$$p(\theta_t \mid Y_t) \propto \exp\{-\frac{1}{2}[(X_t \beta_t - Y_t)^T \Sigma_{\varepsilon_t}^{-1}(X_t \beta_t - Y_t) + (\beta_t - Z_t \hat{\beta}_{t-1})^T M_t^{-1}(\beta_t - Z_t \hat{\beta}_{t-1})]\}.$$

Noting that $p(\theta_t)$ is a natural conjugate prior, we may complete the squares, which is a standard technique in Bayesian inference, to get

$$\theta_t \mid Y_t \sim N[Z_t \hat{\beta}_{t-1} + K_t(Y_t - X_t Z_t \hat{\beta}_{t-1}), M_t - K_t X_t M_t],$$

where

$$K_t = M_t X_t^T (\Sigma_{\varepsilon_t} + X_t M_t X_t^T)^{-1}. \tag{4.6}$$

This, of course, means that

$$\begin{cases} \hat{\beta}_t = Z_t \hat{\beta}_{t-1} + K_t(Y_t - X_t Z_t \hat{\beta}_{t-1}), \\ \hat{\Sigma}_t = M_t - K_t X_t M_t. \end{cases} \tag{4.7}$$

We then proceed with the next iteration. Equations (4.4), (4.6), and (4.7) are known in the literature as the KF. We warn the reader not to confuse the KF with the state-space representation given in (2.1) and (2.3).

By means of various vector-matrix manipulations, the matrix K_t in (4.6) can be put into a number of equivalent forms. An alternative formulation, that we shall use in the next section, is given by

$$K_t = (X_t^T \Sigma_{\varepsilon_t}^{-1} X_t + M_t^{-1})^{-1} X_t^T \Sigma_{\varepsilon_t}^{-1}. \tag{4.8}$$

To verify (4.8), we simply premultiply the right-hand side of (4.6) by a suitable choice of "identity" matrix as follows:

$$\begin{aligned}K_t &= (X_t^T\Sigma_{\varepsilon_t}^{-1}X_t + M_t^{-1})^{-1}(X_t^T\Sigma_{\varepsilon_t}^{-1}X_t + M_t^{-1})M_tX_t^T(\Sigma_{\varepsilon_t} + X_tM_tX_t^T)^{-1} \\ &= (X_t^T\Sigma_{\varepsilon_t}^{-1}X_t + M_t^{-1})^{-1}X_t^T\Sigma_{\varepsilon_t}^{-1}(\Sigma_{\varepsilon_t} + X_tM_tX_t^T)(\Sigma_{\varepsilon_t} + X_tM_tX_t^T)^{-1} \\ &= (X_t^T\Sigma_{\varepsilon_t}^{-1}X_t + M_t^{-1})^{-1}X_t^T\Sigma_{\varepsilon_T}^{-1}.\end{aligned}$$

5. Relationship to Generalized Least Squares

In this section, we briefly discuss the KF relationship with the generalized least squares methodology for both the classical and Bayesian approaches. We suppose that $\beta_1 = \beta_2 = \ldots = \beta_t$, $Z_1 = I$, and Σ_{η_t} does not appear. By simple computations involving the KF with (4.8), we find

$$\begin{aligned}\hat{\beta}_1 &= \hat{\beta}_0 + K_1(Y_1 - X_1\hat{\beta}_0) \qquad (5.1)\\ &= \hat{\beta}_0 + (X_1^T\Sigma_{\varepsilon_1}^{-1}X_1 + \Sigma_0^{-1})^{-1}X_1^T\Sigma_{\varepsilon_1}^{-1}(Y_1 - X_1\hat{\beta}_0) \\ &= (X_1^T\Sigma_{\varepsilon_1}^{-1}X_1 + \Sigma_0^{-1})^{-1}[(X_1^T\Sigma_{\varepsilon_1}^{-1}X_1 + \Sigma_0^{-1})\hat{\beta}_0 + X_1^T\Sigma_{\varepsilon_1}^{-1}(Y_1 - X_1\hat{\beta}_0)] \\ &= (X_1^T\Sigma_{\varepsilon_1}^{-1}X_1 + \Sigma_0^{-1})^{-1}(\Sigma_0^{-1}\hat{\beta}_0 + X_1^T\Sigma_{\varepsilon_1}^{-1}Y_1),\end{aligned}$$

which is the posterior estimate for β_1 when initial information from a natural conjugate prior is available. Notice that when Σ_0^{-1} vanishes (i.e., when the prior is not informative), the estimate is

$$\hat{\beta}_1 = (X_1^T\Sigma_{\varepsilon_1}^{-1}X_1)^{-1}X_1^T\Sigma_{\varepsilon_1}^{-1}Y_1, \qquad (5.2)$$

which is the generalized least squares estimate of β_1.

6. On the Convergence of Rational Expectations

The postulate of the convergence of rationally formed expectations carries with it important economic consequences. In models without sufficient side conditions, the postulate is needed to determine a unique equilibrium at each instant (Muth's model of inventory speculation, 1961, which shows convergence is a very special case of rational expectations models).

In this section, we provide a simple Bayesian sequential method based on the KF to test the convergence of rational expectations by treating the integration constant as an unobserved variable. Our proposal differs from that in Burmeister and Wall (1982) in two main respects: First, we introduce entropy maximizing behavior. Secondly, we state a sequential test, which may save computations when rejection occurs at early stages.

Consider the following simple macroeconomic model consisting of a demand for real money balances as a function of the expected rate of inflation (cf. Cagan's portfolio balance schedule, 1956):

$$m_t - p_t = \gamma(E_t p_{t+1} - p_t) + \delta y_t, \ \gamma < 0, \ \delta > 0, \tag{6.1}$$

and a stochastic money supply given by

$$m_t = \rho m_{t-1} + v_t, \ 0 < \rho < 1, \tag{6.2}$$

where m_t is the natural logarithm of the nominal stock of money at time t, p_t is the natural logarithm of the price level at t, y_t is the natural logarithm of income at t, $E_t p_{t+1}$ is the conditional expectation of p_{t+1} formed at time t and based upon all available information $I_t = \{\gamma, \delta, \rho, m_t, m_{t-1}, \ldots; p_t, p_{t-1}, \ldots\}$, γ is a constant related to the elasticity of the demand for real balances with respect to the expected inflation, δ is a constant related to the elasticity of the demand for real money balances with respect to income, and the shocks, v_t, are independent normal disturbances with mean zero, variance σ_v^2, and $E\{v_t \mid I_t\} = 0$. We make the extreme classical full-employment assumption $y_t = \bar{y}$, a constant. Macroeconomic equilibrium leads to

$$(\lambda - B^{-1}) E_t p_t = \frac{1}{\gamma}(\delta\bar{y} - \rho E_t m_{t-1}),$$

where $\lambda = 1 - \gamma^{-1} > 1$ and the operator B is defined by

$$B^{-j} E_t p_{t+s} = E_t p_{t+s+j}$$

for all $t, s, j \geq 0$. The forward solution to this first order difference equation is

$$p_t = -\frac{\rho}{\gamma\lambda}\sum_{j=0}^{\infty}\lambda^{-j}E_t\, m_{t+j-1} - \delta\bar{y} + \beta_t, \tag{6.3}$$

where β_t is any stochastic process satisfying

$$E\{\beta_{t+1} \mid \hat{I}_t\} = \lambda\beta_t \tag{6.4}$$

with $\hat{I}_t = I_t \cup \{\beta_t, \beta_{t-1}, \ldots\}$. Notice that in such a case

$$(\lambda - B^{-1})E_t\beta_t = \lambda\beta_t - E_t\beta_{t+1} = 0.$$

We may also write β_t as $\beta_t = \lambda^t x_t$, where x_t is any martingale, that is, x_t is any stochastic process that satisfies

$$E\{x_{t+1} \mid \hat{I}_t\} = x_t$$

Therefore, there are infinitely many divergent forward rational expectations solutions. Convergence will require $\beta_t = 0$ for all t.

Furthermore, from successive substitution of (6.2) we can show that $E_t m_{t+j-1} = \rho^j m_{t-1}$, $j = 0, 1, \ldots$, and therefore (6.3) becomes

$$p_t = \frac{\rho m_{t-1}}{1-\gamma(1-\rho)} - \delta\bar{y} + \beta_t. \tag{6.5}$$

There are many stochastic processes (bubbles) consistent with (6.4), for instance,

$$\beta_{t+1} = \begin{cases} \dfrac{\lambda\beta_t}{q} & \text{with prob} q,\ 0 < q \leq 1, \\ 0 & \text{with prob } 1-q, \end{cases}$$

or

$$\beta_{t+1} = \lambda\beta_t + \eta_t, \tag{6.6}$$

where the η_t's are independent Gaussian variables with mean zero and variance σ_η^2.

To be in a position to apply the KF to test convergence, we assume that the β_t's are unobserved variables satisfying (6.6). We also assume that there is partial information in terms of the two first moments on the initial β_0, say, $E\{\beta_0\} = \hat{\beta}_0$ and $E\{\beta_0^2\} = \hat{\sigma}_0^2 + \hat{\beta}_0^2$. If individuals behave as entropy maximizers, then the prior distribution of β_0 compatible with the available distribution is $N(\hat{\beta}_0, \hat{\sigma}_0^2)$. The random variables β_0, ε_t and η_t, as usual, are supposed to be independent. Hence, under normally distributed errors, the rational expectations system is given by

$$\begin{cases} \beta_t = \lambda\beta_{t-1} + \eta_{t-1}, \\ m_t = \rho m_{t-1} + \nu_t, \\ p_t = \dfrac{\rho m_{t-1}}{1 - \gamma(1-\rho)} - \delta\bar{y} + \beta_t, \end{cases}$$

or equivalently, in terms of the transition and measurement equations,

$$\begin{cases} \beta_t = \lambda\beta_{t-1} + \eta_{t-1}, \\ p_t + \delta\bar{y} - \dfrac{m_t}{1 - \gamma(1-\rho)} = \beta_t + \varepsilon_t, \end{cases}$$

where

$$\varepsilon_t \sim N(0, \sigma_\varepsilon^2), \text{ and } \sigma_\varepsilon^2 = \left[\frac{\sigma_\nu}{1 - \gamma(1-\rho)}\right]^2.$$

To test the common assumption of convergence with available data on p_t, m_t, γ, δ, ρ and $\bar{y}$, and under normally distributed errors we refer to the KF, equations (4.4), (4.6) and (4.7) with univariate error terms. In such a case, the posterior distribution of $\beta_t \mid \hat{I}_{t-1}$ is $N(\hat{\beta}_t, \hat{\sigma}_t^2)$, where

$$\begin{cases} \hat{\beta}_t = \theta_t\lambda\hat{\beta}_{t-1} + (\mathrm{i} - \theta_t)\left[p_t + \delta\bar{y} - \dfrac{\rho m_t}{1 - \gamma(1-\rho)}\right], \\ \hat{\sigma}_t^2 = (1 - \theta_t)\sigma_\varepsilon^2, \\ \theta_t = \sigma_\varepsilon^2(\sigma_\varepsilon^2 + \lambda^2\hat{\sigma}_{t-1}^2 + \sigma_\eta^2)^{-1}. \end{cases}$$

The null hypothesis to be tested is $H_0 : \beta_t = 0$ for all $t \geq 1$. Proceeding recursively and starting off at $t = 1$, we reject H_0 if a t appears for which $\beta_t = 0$ does not lie within a highest posterior density interval with a given uniform significance level α, namely,

$$(\hat{\beta}_t - z_{\alpha/2}\hat{\sigma}_t, \hat{\beta}_t + z_{\alpha/2}\hat{\sigma}_t)$$

where, as usual, $P\{Z > z_{\alpha/2}\} = \alpha / 2$ and $Z \sim N(0, 1)$.

7. Relationship with Stochastic Optimal Control

Stochastic optimal control has been very attractive to economists. The literature on its application to economics is abundant. However, the relationship of the KF with stochastic optimal control, given by the separation theorem, has not been fully appreciated. Among the very few papers that have exploited the separation theorem, we mention Sargent (1989) and Basar and Salmon (1989).

In this section we briefly state the separation theorem, and in the next section we apply it to a central planner problem with uncertain inflation.

Let us extend our considerations to the dynamic system

$$\begin{cases} Y_t = X_t\beta_t + \varepsilon_t, \\ \beta_t = Z_t\beta_{t-1} + L_t u_{t-1} + \eta_{t-1}, \end{cases} \tag{7.1}$$

with finite horizon $t = 1, \ldots, N$. When the control term $L_t u_{t-1}$ is added, it is natural to ask that the composite system of control and estimation be jointly optimal in some well-defined sense. We shall be particularly concerned with the case when the dynamic linear system has quadratic performance indices. Thus, the optimal control will be determined as a solution to

$$\text{Minimize } E\left\{\sum_{t=1}^{N} \beta_t^T P_t \beta_t + u_{t-1}^T Q_{t-1} u_{t-1}\right\}, \tag{7.2}$$

$$\text{subject to: } \begin{cases} Y_t = X_t\beta_t + \varepsilon_t, \\ \beta_t = Z_t\beta_{t-1} + L_t u_{t-1} + \eta_{t-1}, \\ (u_0, \ldots, u_{N-1}) \in U, \end{cases}$$

where U is a specified set, P_t and Q_t are symmetric matrices, P_t is semipositive-definite, Q_t is positive-definite, and β_0 and the noise terms ε_t and η_t are, as before, assumed to be independent.

The above problem without the measurement constraint, $Y_t = X_t\beta_t + \varepsilon_t$, is known in the stochastic optimal control literature as the discrete-time stochastic linear optimal regulator problem or the discrete-time LQG (Linear-Quadratic-Gaussian) problem (see Venegas, 1992b and 1993).

By using stochastic dynamic programming (Bellman's recursive equations, 1957) to characterize the optimal control of problem (7.2), we can show that a necessary condition for u_t to be a minimum is that

$$\hat{u}_t = \Lambda_t\hat{\beta}_t, \tag{7.3}$$

where the guidance or control matrix Λ_t satisfies

$$\begin{cases} \Lambda_t = -[L_{t+1}^T W_{t+1} L_{t+1} + Q_t]^{-1} L_{t+1} W_{t+1} X_{t+1}, \\ W_t = Z_{t+1}^T W_{t+1} Z_{t+1} + Z_{t+1}^T W_{t+1} L_{t+1} \Lambda_t + P_t, \\ P_N = W_N, \end{cases}$$

and $\hat{\beta}_t$ satisfies the KF, that is,

$$\begin{cases} M_t = Z_t \hat{\Sigma}_{t-1} Z_t^T + \Sigma_{\eta t-1}, \\ K_t = M_t X_t^T (\Sigma_{\varepsilon_t} + X_t M_t X_t^T)^{-1}, \\ \hat{\Sigma}_t = M_t - K_t X_t M_t, \\ \hat{\beta}_t = Z_t \hat{\beta}_{t-1} + K_t (Y_t - X_t Z_t \hat{\beta}_{t-1}). \end{cases}$$

The above result is known in the literature as the separation theorem since Λ_t is obtained as solution to the deterministic problem (see, for instance, Meditch, 1969).

8. A Central Planner Problem under Uncertain Inflation

In this section we use the separation theorem in studying a central planner problem under uncertain inflation. We consider a one-good economy without capital resources.

8.1. *Private sector*

We suppose there is a large number of identical consumers, each of whom makes consumption decisions in $T-1$ periods ($t=0, 1, \ldots, T-1$), and has the following budget constraint:

$$w_{t-1}M_t = w_{t-1}M_{t-1} + g_{t-1} + y_{t-1} - c_{t-1}, \quad t = 1, \ldots, T, \tag{8.1}$$

$$M_0 > 0 \text{ given}, \quad M_T > 0,$$

where M_t is the stock of currency owned at the beginning of period t, w_t is the value of the currency measured in goods at t (the reciprocal of the price level), g_t stands for government lump-sum transfers at t, y_t is real income at t, and c_t is consumption at t. Equation (8.1) can be rewritten, in terms of the inflation rate

$$\pi_t = \frac{w_{t-1}}{w_t} - 1$$

as

$$(1+\pi_t)m_t = (1+\pi_{t-1})m_{t-1} + g_{t-1} + y_{t-1} - c_{t-1} - \pi_{t-1}m_{t-1}, \tag{8.2}$$

$$t = 1, \ldots, T,$$

where $m_t = w_t M_t$ represents real money balances and the last term on the right-hand side stands for depreciation of real money balances from inflation. Notice, however that the above budget constraint requires additional information on w_{-1} and w_T,

Private agents have no knowledge of $w_{-1}, w_0, \ldots, w_T$, and therefore, they do not know the inflation rate, π_t. However, we assume they have partial information on the distribution of w_{-1}, in terms of the first two moments, say, $E\{w_{-1}\} = \hat{w}_{-1}$ and

$$E\{w^2_{-1}\} = \hat{\sigma}^2_{-1} + \hat{w}^2_{-1}.$$

Assuming that individuals behave as entropy maximizers, then the prior distribution for w_{-1} that is compatible with the available information is $N(\hat{w}_{-1}, \hat{\sigma}^2_{-1})$. Therefore,

$$w_{-1}M_0 = (1+\pi_0)m_0 \sim N(\hat{w}_{-1}M_0, \hat{\sigma}^2_{-1}M_0^2).$$

Of course, we assume that $\hat{w}_{-1} > 0$.

Suppose also that private agents are capable of making indirect measurements, $\bar{\pi}_t$, of π_t, according to the rule

$$(1+\bar{\pi}_t)\bar{m} = (1+\pi_t)m_t + \varepsilon_t, \quad t = 1, \ldots, T, \tag{8.3}$$

where, as in Tabellini (1986), $\bar{m}$ is a constant target chosen by the monetary authority at $t = 1$. We assume that the observation errors, ε_t, are independent normal random variables with mean zero, variance σ^2_ε and $E\{w_{-1}\varepsilon_t\} = 0$.

The representative individual's objective is to maximize, at the present ($t = 0$), his total expected utility of consumption over $T - 1$ periods, namely,

$$E\left\{\sum_{t=1}^{T} u(c_{t-1}) + \Psi(m_T)\right\}. \tag{8.4}$$

Notice that, for simplicity, no discount factor has been included in the overall utility, and money services provide no utility. The utility function is expressed as the quadratic function

$$u(c_t) = a_1 c_t - \frac{a_2}{2} c_t^2, \quad t = 0, \ldots, T-1. \tag{8.5}$$

Here, $a_1, a_2 > 0$, and the ratio a_1 / a_2 determines the level of satiation. Notice that $u(0) = u(2a_1 / a_2) = 0$, $u(c_t) > 0$ for $0 < c_t < 2a_1 / a_2$, $u(c_t) < 0$ for $c_t > 2a_1 / a_2$, $u'(c_t) \geq 0$ for $0 \leq c_t \leq a_1 / a_2$, and $u'(c_t) < 0$ for $c_t > a_1 / a_2$. The salvage value is chosen as

$$\Psi(m_T) = -(a_2 / 2)[(1 + \pi_T)m_T]^2.$$

We assume that the real income of the individual randomly fluctuates around his income satiation level following

$$y_t = \frac{a_1}{a_2} + \eta_t, \ \eta_t \sim N(0, \sigma_\eta^2), \ t = 0, \ldots, T-1, \tag{8.6}$$

where the η_t's are independent endowment shocks satisfying $E\{\varepsilon_s \eta_t\} = 0$ for all t, s, and $E\{w_{-1}\eta_t\} = 0$.

8.2. *Public sector*

In order to keep monetary experiments as separate as possible from the effect of other government activities, we suppose that at each time $t = 0, 1, \ldots, T-1$, the government consumes nothing, has no debt and is committed to provide a lump-sum subsidy to compensate for depreciation of real money balances whatever the rate of inflation is. Thus, the government budget constraint is given by (cf. Calvo, 1991)

$$g_t = \pi_t m_t, \quad t = 0, \ldots, T-1. \tag{8.7}$$

8.3. *The command optimum*

After incorporating the government behavior, (8.7), and the real income fluctuations, (8.4), into the representative individual's budget constraint, (8.2), we get the consolidated constraint for the economy

$$(1 + \pi_t)m_t = (1 + \pi_{t-1})m_{t-1} - (c_{t-1} - \frac{a_1}{a_2}) + \eta_{t-1}, \quad t = 1, \ldots, T. \tag{8.8}$$

Let us denote $\beta_t = (1 + \pi_t)m_t$, $\hat{\beta}_0 = \hat{w}_{-1}M_0$ and $\hat{\sigma}_0^2 = \hat{\sigma}_{-1}^2 M_0^2$. Notice that π_t is unobserved, and therefore β_t is unobserved. The social planning problem is thus stated as

$$\text{Minimize } E\left\{\sum_{t=1}^{T}\left(c_{t-1}-\frac{a_1}{a_2}\right)^2+\beta_T^2\right\},$$

subject to:

$$\begin{cases} \beta_t=\beta_{t-1}-(c_{t-1}-\dfrac{a_1}{a_2})+\eta_{t-1}, \quad t=1,\ldots,T, \\ (1+\bar{\pi}_t)\bar{m}=\beta_t+\varepsilon_t, \quad t=1,\ldots,T, \\ \beta_0 \sim N(\hat{\beta}_0, \hat{\sigma}_0^2), \\ \varepsilon_t \sim N(0,\sigma_\varepsilon^2),\ \eta_t \sim N(0,\sigma_\eta^2), \text{ with } \beta_0, \varepsilon_t \text{ and } \eta_t \text{ independent.} \end{cases} \tag{8.9}$$

Constraints (8.9) determine the state-space representation of the dynamics of β_t with control c_{t-1}. It is worthwhile to note that the constraints (8.9) collapse into $\tilde{y}_{t-1}=c_{t-1}$, where $\tilde{y}_{t-1}=y_{t-1}+\xi_{t-1}-\xi_t$, $\xi_0=\beta_0-\bar{m}$, $\xi_t \sim N(\bar{\pi}_t\bar{m},\sigma_\varepsilon^2)$ for $t=1,\ldots,T-1$, and $\xi_T=\beta_T-\bar{m}$. According to the separation theorem stated in (7.3), the optimal planned consumption path, $\{\hat{c}_t\}_{t=0}^{T-1}$, satisfies

$$\hat{c}_t=\frac{a_1}{a_2}+\frac{1}{T-t+1}\hat{\beta}_t, \quad t=0,\ldots,T-1, \tag{8.10}$$

where the estimates $\hat{\beta}_t$ are computed through the KF, equations (4.4), (4.6) and (4.7) with univariate error terms, as

$$\hat{\beta}_t=\theta_t\hat{\beta}_{t-1}+(1-\theta_t)(1+\bar{\pi}_t)\bar{m}, \quad t=1,\ldots,T-1, \tag{8.11}$$

$$\theta_t=\frac{\sigma_\varepsilon^2}{\sigma_\varepsilon^2+\hat{\sigma}_{t-1}^2+\sigma_\eta^2}, \quad t=1,\ldots,T-1, \tag{8.12}$$

$$\hat{\sigma}_t^2=(1-\theta_t)\sigma_\varepsilon^2, \quad t=1,\ldots,T-1. \tag{8.13}$$

Moreover, the optimal salvage value is reached at

$$\hat{\beta}_T=\theta_T\hat{\beta}_{T-1}+(1-\theta_T)(1+\bar{\pi}_T)\bar{m}>0.$$

From above, we have the following results for the optimal consumption path.

PROPOSITION8.1. *In order to overtake uncertain inflation the centrally planned consumption will exceed the satisfaction level at all times.*

PROOF. Since $\hat{w}_{-1} > 0$ and $M_0 > 0$, then $\hat{\beta}_0 > 0$. Therefore, from the recursive property of $\hat{\beta}_t$ stated in (8.11) and since $\theta_t > 0$ for all t, we obtain $\hat{\beta}_t > 0$ for all t. Thus, equation (8.10) implies

$$\hat{c}_t - \frac{a_1}{a_2} = \frac{1}{T-t+1}\hat{\beta}_t > 0, \quad t = 0, \ldots, T-1.$$

Comparative statics

We are now concerned with the effects on optimal consumption, $\hat{c}_t$, when the parameters of the initial distribution are changed.

PROPOSITION8.2. *Under uncertain inflation we have the following effects on the centrally planned consumption:*

(i) An increase in the initial mean value of the currency, $\hat{w}_{-1}$, will increase the centrally planned consumption at all times.

(ii) The effect on consumption due to a change in the initial variance, $\hat{\sigma}^2_{-1}$, is undetermined at all times.

(iii) The effect on consumption due to a change in the initial stock of money, M_0, is undetermined at all times.

PROOF. *(i)* Observe first that from (8.10) and (8.11)

$$\frac{\partial \hat{c}_t}{\partial \hat{\beta}_0} = \frac{1}{T-t+1}\prod_{j=1}^{t}\theta_{t-j+1} > 0, \quad t = 0, \ldots, T-1,$$

so

$$\frac{\partial \hat{c}_t}{\partial \hat{w}_{-1}} = \frac{\partial \hat{c}_t}{\partial \hat{\beta}_0} M_0 > 0, \quad t = 0, \ldots, T-\mathrm{i}.$$

(ii) Notice that $\partial \hat{c}_0 / \partial \hat{\sigma}^2_{-1} = 0$ and

$$\frac{\partial \hat{c}_t}{\partial \hat{\sigma}^2_0} = \frac{1}{T-t+1}\{A_t B_t + \theta_t[A_{t-1}B_{t-1} + \theta_{t-1}[\ldots[A_1 B_1]\ldots]]\}, \qquad (8.14)$$

$$t = \mathrm{i}, \ldots, T-\mathrm{i},$$

where $A_t = \hat{\beta}_{t-1} - (1 + \bar{\pi}_t)\bar{m}$ and $B_t = \partial\theta_t / \partial\hat{\sigma}_0^2$. We readily see that

$$B_t = \underbrace{\left(\frac{\partial\theta_t}{\partial\hat{\sigma}_{t-1}^2}\frac{\partial\hat{\sigma}_{t-1}^2}{\partial\theta_{t-1}}\right)}_{(+)}\underbrace{\left(\frac{\partial\theta_{t-1}}{\partial\hat{\sigma}_{t-2}^2}\frac{\partial\hat{\sigma}_{t-2}^2}{\partial\theta_{t-2}}\right)}_{(+)}\cdots\underbrace{\left(\frac{\partial\theta_1}{\partial\hat{\sigma}_0^2}\right)}_{(-)} < 0,\ t = 1, \ldots, T-1$$

since, from (8.12) and (8.13), we get

$$\frac{\partial\theta_t}{\partial\hat{\sigma}_{t-1}^2} < \theta, \text{ and } \frac{\partial\hat{\sigma}_{t-1}^2}{\partial\theta_{t-1}} < \theta.$$

However, the A_t's may be of either sign, and therefore $\partial\hat{c}_t / \partial\hat{\sigma}_0^2$, in (8.14), cannot be signed. Hence,

$$\frac{\partial\hat{c}_t}{\partial\hat{\sigma}_{-1}^2} = \frac{\partial\hat{c}_t}{\partial\hat{\sigma}_0^2}M_0^2,\quad t = 0, \ldots, T-1,$$

has ambiguous sign at all times.

(iii) In this case,

$$\frac{\partial\hat{c}_t}{\partial M_0} = \frac{\partial\hat{c}_t}{\partial\hat{\beta}_0}\hat{w}_{-1} + \frac{\partial\hat{c}_t}{\partial\hat{\sigma}_0^2}2M_0\hat{\sigma}_{-1}^2,\quad t = \theta, \ldots, T-1. \tag{8.15}$$

On the right-hand side of (8.15) we have two effects: The *mean* effect

$$\left.\frac{\partial\hat{c}_t}{\partial M_0}\right|_{\hat{\sigma}_0^2 = constant} = \frac{\partial\hat{c}_t}{\partial\hat{\beta}_0}\hat{w}_{-1}$$

which, from *(i)*, is definitely positive, and the *variance* effect

$$\left.\frac{\partial\hat{c}_t}{\partial M_0}\right|_{\hat{\beta}_0 = constant} = \frac{\partial\hat{c}_t}{\partial\hat{\sigma}_0^2}2M_0\hat{\sigma}_{-1}^2$$

which, from *(ii)*, is ambiguous. This concludes the proof of Proposition 8.2.

Some comments are in order: Proposition 8.1 and *(i)* in Proposition 8.2 seem to be very intuitive. However, in Proposition 8.2, at any time during the planning horizon, the variance effect, in *(ii)*, may be of either sign, and the overall impact, in *(iii)*, will depend on which effect dominates. If the mean effect is large enough, the total change in consumption could be positive.

9. Summary and Conclusions

We have presented the KF in a way that might be attractive to economists by using information theory and Bayesian inference. The recursive updating of the KF was developed as closely as possible to that of the sequential learning mechanism used by Lucas (1973). We have discussed the relationship of the KF with some models in the econometric literature, including the generalized least squares, under both the classical and the Bayesian frameworks. We emphasized, throughout the paper, the potential use of the KF in modeling economic systems. We provided a Bayesian sequential test on the convergence of rational expectations, and studied a central planner problem under uncertain inflation. We are aware that more work has to be done on possible extensions, including the nonnormal state-space model with correlated noises, to get more results for the proposed sequential test and for the central planner problem.

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