EQUILIBRIUM BEHAVIOR IN ALL-PAY AUCTION WITH COMPLETE INFORMATION

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Resumen: En una subasta todos-pagan, los licitantes pagan el precio ofrecido y el licitante que somete la oferta más alta gana el bien. Los equilibrios de Nash de este juego incluyen el uso de estrategias aleatorias, que protegen a los licitantes de perder por una pequeña cantidad. Este documento generaliza el análisis ordinario para permitir "errores" de decisión endógenos, que pueden ser debido a equivocaciones o a variaciones aleatorias no observadas en las funciones de pago. La distribución de los errores depende de los pagos esperados en equilibrio, que a su vez determinan las distribuciones de errores como un punto fijo. Un resultado derivado de este trabajo es que el equilibrio generalizado de Nash y el equilibrio Nash de la subasta todos-pagan son equivalentes si los términos de error son idéntica e independientemente distribuidos.

Abstract: A widely used sealed-bid auction is the first-price auction. In this auction, the highest bidder wins the item and pays the price submitted; the other bidders get and pay nothing. The all-pay auction is similar to the first-price auction, except that losers must also pay their submitted bids. The Nash equilibria of this game involve the use of randomized strategies, which protect bidders from being overbid by a small amount. This paper generalizes the standard Nash equilibrium analysis of the all-pay auction to allow for endogenously determined decision "errors". Such errors may either be due to mistakes or to unobserved random variation in payoff functions. The error distributions depend on equilibrium expected payoffs, which in turn determine the error distributions as a fixed point. A striking result derived in this paper is that for any structure of the error terms the generalized Nash equilibrium and the Nash equilibrium of the all-pay auction are equivalent if the error terms are identically and independently distributed.

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1. Introduction

Auctions are one of the basic mechanisms for determining the prices of goods to be exchanged. In auctions, prices are determined by competition among potential buyers. Since the price in an auction is determined when the object is sold, it reflects all the available information and the preferences of the potential buyers who are bidding. Auctions may take one of two basic forms, oral or sealed-bid. In oral auctions, bidders hear one another's bids as they are each made. In sealed-bid auctions, bidders simultaneously submit one or more bids to the seller without revealing their bids to one another.[1] A widely used sealed-bid auction is the first-price auction. In this auction, the highest bidder wins the item and pays the price submitted; the other bidders get and pay nothing.

The all-pay auction is similar to the first-price auction, except that losers must also pay their submitted bids. Baye, Kovenock and De Vries (1995) fully characterize the set of Nash equilibria in the first-price all-pay auction with complete information. In contrast to previous research, they show that the set of equilibria is much larger than the set of symmetric equilibria.

Many economic problems can be modeled with the all-pay auction. In situations such as lobbying for rents in regulated or protected industries, technological competition and political campaigns, the participant showing the greatest effort or expenditure wins the prize, while the others are penalized. For example, Dasgupta (1986) uses the all-pay auction to model patent races in which the bids are research and development expenditures and the prize is a patent with known value. The firm spending the most on research and development obtains the patent, while the other firms make loses since they do not recover their expenditures.[2]

A characteristic of this model is that the reward structure is such that ex-post payoffs are discontinuous. This property precludes the existence of Nash equilibrium in pure strategies.[3] The Nash equi-

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[1] For a further discussion of auctions, see, for example, McAfee and McMillan (1987), Milgrom and Weber (1982) and Myerson (1991).

[2] Moulin (1986) also examines this symmetric equilibrium, but interprets it as a lobbying game.

[3] Dasgupta and Maskin (1982) have shown that discontinuous games do possess mixed-strategy Nash equilibria under certain restrictions. A sufficient set of conditions is that the firm's profit function is everywhere left lower semi-continuous in its price, (and hence weakly lower semi-continuous), the profit function is bounded, and the sum of the two firms' profit functions is continuous.

librium of the all-pay auction with complete information typically involves the use of randomized strategies, which protect bidders from being overbid by a small amount. Experimental data seems to track the qualitative features of Nash equilibria in games with structures similar to the all-pay auction, but prices are often much higher than the equilibrium predictions (Davis and Holt, 1994; Kruse *et al.*, 1994, and Baye and Morgan 2002).[4]

In order to sort out the reasons for the observed departures from the Nash prediction, a useful positive theory of behavior in games could begin by qualifying the assumption that individuals are perfect maximizers of their own money payoffs. Several authors have relaxed the perfect rationality assumption in experimental games: Brown and Rosenthal (1990), Camerer and Weigelt (1990), McKelvey and Palfrey (1992, 1995), Banks *et al.* (1994), Brandts and Holt (1992), Palfrey and Rosenthal (1991, 1992), and Baye and Morgan (2002). One way is to introduce decision error, i.e. in choosing their strategies players make mistakes.

As a first step, it is useful to distinguish two sources of deviations from the Nash equilibria as calculated from expected money payoffs. First, *systematic* deviations may be due to the importance of neglected factors, such as altruism, envy, fairness, etc. Second, *nonsystematic* or random "errors", can follow from mistakes in recording decisions, from time constraints as in chess games, or from random errors in evaluating small differences in expected payoffs. Experimental evidence suggests that nonsystematic errors can occur in strategic situations (McKelvey and Palfrey, 1993) and also in simpler individual decision-making tasks (Anderson, 1994).

This paper investigates the quantal response equilibrium of the all-pay auction model in which boundedly rational players may interact. In contrast to the classical conception of rationality that is based on unlimited capacity, boundedly rational players are limited by their own computational ability. Boundedly rational players have been most commonly characterized by either the random choice or the random utility version of discrete choice theory. The discrete choice framework is used to analyze the strategic interaction of multiple individuals. The quantal response equilibrium is a game theoretic equilibrium concept. In this model, players choose among strategies

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[4] The experiment procedures in such work involved 6 two-hour sessions, each with a different cohort of five subjects. Each session lasted for 60 periods. Participants were undergraduate business students with previous experience at their own terminals, the instructions were read aloud to them by the experimenter. Participants were paid in cash.

based on their expected payoffs, but make decision errors based on a quantal or probabilistic choice model, and assume other players do so as well. Such errors may either be due to mistakes or to unobserved random variations in payoff functions. For a given error distribution, a quantal response equilibrium is a fixed point in choice probabilities.

The main result derived in this paper is that with continuous or with discrete bid choices the symmetric Nash equilibrium in mixed-strategies and quantal response equilibrium of the all-pay auction turn out to be identical if the error terms are identically and independently distributed. In addition, this paper calculates the quantal response equilibria of the all pay auction for two particular parametrizations: the power-function and the logit equilibrium. The theoretical results obtained in this paper have clear implications for experimental research and have been used to suggest designs for further experiments (Baye and Morgan 2002).

This paper consists of two parts. In the first part, the Nash equilibrium of the (first-price) all-pay auction is analyzed. Section 2 contains the model. Following Dasgupta (1986), section 3 examines the all-pay auction with continuous bid choices. In many laboratory experiments, bids are constrained to integer values (e.g. pennies); the calculation of the symmetric Nash equilibrium in mixed-strategies for discrete bid choices is the topic of section 4. Section 5 introduces the equilibrium concept. The second part of this paper examines the quantal response equilibrium of the all-pay auction. In sections 6 and 7, the models presented in sections 2 and 3 are generalized to allow decisions errors. It is shown that for any common structure of the error term the symmetric Nash equilibrium in mixed-strategies and quantal response equilibrium of the all-pay auction are identical. Importantly, this result holds only if the error terms are identical and independently distributed. Section 8 calculates the quantal response equilibria of the all-pay auction for two particular parametrizations: the power-function and the logit equilibrium. The power-function equilibrium is based on random utility maximization with multiplicative error terms while the logit equilibrium is derived from random utility maximization with additive error terms. Section 9 contains concluding remarks. The appendix derives the quantal response equilibria of the all-pay auction with multiplicative error terms.

2. The Model

Assume that there are 2 identical bidders (henceforth referred to as firms). The "bids", which can be interpreted as competitive expenditures, are set simultaneously. The firm spending the most obtains the prize. Each of these firms has an identical known valuation, v. For instance, in a research and development (R&D) race, the bids are R&D expenditures and the prize is a patent with corresponding monopoly profit. In political contests, the bids are lobbying expenditures and the prize is a political favor. The value is split in case of a tie. The payoff to firm 1 is given by:

$$\pi_1(p_1) = \begin{cases} v - p_1 & \text{for} \quad p_1 > p_2 \\ \frac{v}{2} - p_1 & \text{for} \quad p_1 = p_2 \\ -p_1 & \text{for} \quad p_1 < p_2 \,. \end{cases} \tag{1}$$

Notice from (1) that the bid pi is paid whether or not the prize is won. Competition in an all-pay auction may be risky because it can generate negative profits. For such a bidding contest to take place, its outcome cannot be deterministic. Each player must have at least some chance of winning in order to be willing to participate.

3. The Nash Equilibrium with Continuous Bid Choices

Let p_i denote the bid posted by firm i, i=1, 2. Notice from (1) that at a bid p_i above v, the firm with the higher bid makes negative profits. Also, no firm is allowed to set a bid p_i less than 0. The calculation of the Nash equilibrium for the all-pay auction typically involves mixed strategies. To see this, suppose there is a pure strategy equilibrium with firm 1 bidding p_1 and firm 2 bidding p_2 and where:

$$p_1 > p_2 \geq 0$$

Now, consider the bid p_1':

$$p_1' = (p_1 + p_2)/2$$

Since $v - (p_1 + p_2)/2 > v - p_1$, firm 1 wins the prize and earns higher profits from bidding p_1' than p_1. Suppose $p_1 = p_2 \geq 0$. Then, a firm by raising slightly its bid wins the prize for sure. Using this argument, it follows that there is not a pure?strategy Nash equilibrium in the all-pay auction with complete information.

3.1. *Nash Equilibrium of the Symmetric All-Pay Auction*

PROPOSITION 1. *In the symmetric-mixed strategy Nash equilibrium, each firm bids randomly with probability 1/v in the support [0, v]. Furthermore, expected profits are zero in equilibrium.* (Dasgupta 1986, p. 536)

Next, the symmetric mixed-strategy distribution with support $[\underline{p}, \bar{p}]$ is constructed. There can be no mass points in the interval $[\underline{p}, \bar{p}]$. The reason is that if there were mass points in this interval, it would pay for a rival to concentrate just below such a mass point, to increase its payoff.

Since there are no mass points in the equilibrium density, $f(p)$, the equilibrium cumulative distribution function, $F(p)$, will be a continuous function on $[\underline{p}, \bar{p}]$. The possibility of ties is not considered until the next section, where bids are integer-valued. Notice that when a firm bids p, it may be that p is the highest bid posted, in which case, the firm's profit is $v - p$. This happens only if the other firm bids lower than p, an event which has probability $F(p)$. Thus the firm's expected profit function is $vF(p) - p$. For firm 1 to be indifferent between bidding some arbitrary bid p and 0, it must be the case that firm 2 bids according to a distribution $F_2(p)$ that makes firm 1's expected earnings at p equal to a security expected profit, S_1. Otherwise, it would pay a firm to increase the frequency for the bid with the higher expected payoff. The equilibrium expected payoff S_1, must satisfy:

$$S_1 = vF_2(p) - p \quad \text{for} \quad p \in (\underline{p}, \bar{p}) \tag{2}$$

In a symmetric equilibrium, $F_1(p) = F_2(p)$, and in this case, (2) yields:

$$F(p) = \frac{p + S}{v} \tag{3}$$

Equation (3) determines the equilibrium distribution function, once the S constant is found from an analysis of boundary conditions, which is the next task. Recall that there are no mass points in this equilibrium. Given $F(\underline{p}) = 0$, it follows that $S = -\underline{p}$ in equation (2). Since bidding zero is a permissible strategy in this model, it must be the case that $\underline{p} = 0$ and hence $S = 0$. Given $F(\bar{p}) = 1$, equation (2) can be used to show that $v - \bar{p} = 0$ so $v = \bar{p}$. Using the boundary conditions, $F(0) = 0$ and $F(v) = 1$, yields the equilibrium probability distribution. It follows that the bidder must bid so that:

$$F(p) = \frac{p}{v} \tag{4}$$

The corresponding price density is written as:

$$f(p) = \frac{1}{v} \tag{5}$$

To summarize, (4) specifies the mixed distribution that a bidder must use for the other bidder to be willing to choose randomly in a range of bids (which by construction, yield equal expected profits). Given symmetry, the other bidder must also use the bid distribution in (4) for the other to randomize. In equilibrium, both bidders randomize according to (4). Finally, the equilibrium distribution is bounded between 0 and v. Also, in equilibrium, the expected payoff to each player is 0. Importantly, this result does not hold in the asymmetric case.

3.2. *Nash Equilibrium of the Asymmetric All-Pay Auction*

We now consider the all-pay auction with $v_1 > v_2$. This case is economically interesting because in the literature of regulation (Rogerson, 1982) and political contests (Snyder, 1989), one player (often the incumbent) is modeled as having an advantage over a challenger. In this game for firm 1 to be indifferent between bidding some arbitrary bid p and 0, it must be the case that firm 2 bids according to a distribution $F_2(p)$ that makes firm 1's expected earnings at p equal to a security expected profit, S_1. Otherwise, it would pay a firm to increase the frequency for the bid with the higher expected payoff. Hence the equilibrium expected payoffs S_1 and S_2, must satisfy:

$$S_1 = v_1 F_2(p) - p \quad \text{for} \quad p \in (\underline{p}, \bar{p}) \tag{6}$$

$$S_2 = v_2 F_1(p) - p \quad \text{for} \quad p \in (\underline{p}, \bar{p})$$

This yields:

$$F_1(p) = \frac{p + S_2}{v_2} \tag{7}$$

$$F_2(p) = \frac{p + S_1}{v_1}$$

The equations in (7) determine the equilibrium distribution functions, once the constants S_1 and S_2 are found from an analysis of boundary conditions, which is the next task. First, note that the lower bound of the equilibrium price distribution is different for both bidders, $F_2(\underline{p}) = (v_1 - v_2)/v_1$ and $F_1(\underline{p}) = 0$. From (7) we have that $S_2 = -\underline{p}$. Since bidding zero is a permissible strategy in this model, it must be the case that $\underline{p} = 0$ and so $S_1 = v_1 - v_2$. Given $F_1(\bar{p}) = 1$, the first equation in (7) can be used to show that $v_2 - \bar{p} = 0$ therefore $v_2 = \bar{p}$. Thus, the boundary conditions $F_1(0) = 0, F_2(0) = (v_1 - v_2)/v_1, F_1(v_2) = 1$ and $F_2(v_2) = 1$ yield the equilibrium probability distributions:

$$F_1(p) = \frac{p}{v_2} \tag{8}$$

$$F_2(p) = \frac{p + v_1 - v_2}{v_1}$$

It follows that bidder 1 and bidder 2 randomize according to (8) over the interval $[0, v_2]$. In equilibrium bidder one earns an expected payoff of $v_1 - v_2$, while the second bidder earns an expected payoff of zero. In contrast to the symmetric all-pay auction, where the expected payoff to the auctioneer (the sum of the expected bids) is v, in the asymmetric case the expected sum of the bids is $v_2^2/v_1 + ((v_1 + v_2) * v_2)/v_1^2$.[5]

To summarize:

PROPOSITION 2. *In the Nash equilibrium of the asymmetric all-pay auction, firm 1 bids randomly with probability* $1/v_2$ *in the support* $[0, v_2]$, *while firm 2 bids randomly with probability* $1/v_1$ *in the support* $[0, v_2]$. *Furthermore, expected profits are different in equilibrium.*

4. The Nash Equilibrium with Discrete Bid Choices

The rules of laboratory experiments typically require that bidders (firms) post bids in pennies. Therefore, the set of feasible decisions

[5] Suppose 3 players and $v_1 > v_2 = v_3$. In the asymmetric equilibrium, player 3 bids zero with probability one, while $F_1(p) = p/v_2$ and $F_2(p) = (v_2 + p)/v_1$ on $[0, v_2]$. There is another equilibrium where player 1 randomizes with $F_1(p) = p[(v_2 + p)/v_1]^{-1/2}$, while players 2 and 3 randomize with $F_2(p) = F_3(p) = [(v_2 + p)/v_1]^{1/2}$.

is finite. The analysis of the mixed equilibria when the decision space is discrete is discussed in game theory texts. Moulin (1986) illustrates the calculation of mixed strategies for a number of simple examples, for example bimatrix games. Davis and Holt (1990) and López-Acevedo (1995) calculate mixed strategy Nash equilibrium with discrete bid choices for a number of games used in laboratory settings. The bid-choice games discussed in this section are much more difficult in the sense that the range of randomization must be determined. The intuition gained from previous section can be useful.

4.1. *Nash Equilibrium of the Symmetric All-Pay Auction*

The calculation of the mixed-strategy Nash equilibrium when bids are restricted to be integer-valued is similar to the one for the continuous case. The equilibrium expected payoff S is given in equation (9). *This equation is comparable to equation (2) but equation (9) also includes the payoff function that determines earnings when a firm's bid matches the other's bid.* The density, $f(p_i)$, denotes the equilibrium probability that a price selected is p_i, where $f(p_i) \geq 0$ for $p_i = p_1, \ldots, v$:

$$S = \left[\sum_{p_i=p_1}^{p_{k-1}} f(p_i)\right](v - p_k) + f(p_k)(\frac{v}{2} - p_k) \tag{9}$$

$$+ \left[1 - \sum_{p_i=p_1}^{p_{k-1}} f(p_i)\right](-p_k)$$

where $p_k = p_1, \ldots, v$. The first term in (9) is the expected profit from being the higher bidder. The second term is the expected profit of a tie at p_k. At this bid, the prize is divided equally. The last term corresponds to the expected payoff from being outbid. Equation (9) can also be expressed as:

$$S = \left[\sum_{p_i=p_1}^{p_k-1} f(p_i) + \frac{f(p_k)}{2}\right](v - p_k) \tag{10}$$

$$+ \left[1 - \left[\sum_{p_i=p_1}^{p_k-1} f(p_i) + \frac{f(p_k)}{2}\right]\right](-p_k)$$

The $G(p_k)$ in equation (11) is a *modified* "distribution function" that allows for the event of ties:

$$G(p_k) = \sum_{p_i=p_1}^{p_{k-1}} f(p_i) + \frac{f(p_k)}{2} = \frac{p_k + S}{v} \tag{11}$$

where the final equality follows from (10). In order to obtain the support of the equilibrium mixed-strategy Nash equilibrium, consider a set of consecutive integer-valued bids: $[p_1, p_2, ..., v]$, where v is the largest integer bid. Define p_L and p_H as the lowest and highest bids respectively that are selected with strictly positive probability, where $p_1 \leq p_L < p_H \leq v$. By evaluating (10) at p_H and using the fact that the sum of the densities up to $f(p_H)$ equals one, one obtains:

$$\begin{aligned} S &= [1 - f(p_H)]\,(v - p_H) + f(p_H)(\frac{v}{2} - p_H) \\ &= v - p_H - f(p_H)\frac{v}{2} \end{aligned} \tag{12}$$

Since $f(p_H) > 0$, it follows from (9) that $S < v - p_H$. Now, we calculate the mixed strategy equilibrium for this model. Conjecture that $f(p_k) = 1/v$, with the upper bound $p_H = v - 1$ and the lower bound $p_1 = p_L = 0$, is the symmetric Nash equilibrium in mixed-strategies.[6] Next, we verify that a seller is indifferent between the bids $0, 1, ..., v - 1$. By evaluating (10) at $p_H = v - 1$, one obtains:

$$\begin{aligned} S &= \left[F(v-2) + \frac{f(v-1)}{2}\right] v - (v-1) \\ &= \left[\frac{v-1}{v} + \frac{1}{2v}\right] v - (v-1) \end{aligned} \tag{13}$$

It is straightforward to verify from (13) that $S = 1/2$. An analogous argument shows that at the lower bound, $p_L = 0, S = 1/2$, and similarly for intermediate prices. From equation (9), it follows that in the event of ties the prize is divided equally. In contrast to the Nash equilibrium with continuous bid choices, rents are not dissipated in equilibrium in the discrete case. A possible reason is that in the discrete case a firm has to bid higher money amounts to outbid a rival.

[6] Solving for $f(p_H)$, it follows that individual 2 must price so that $f(p_H) = \frac{2(v-p_H-S)}{v}$. This specifies the mixed distribution that individual 2 must use in order for individual 1 to be willing to choose randomly in a range of bids.

Furthermore, in equilibrium, the expected payoff to the auctioneer (the sum of the expected bids) equals $(v-1)+1/2$.

To summarize:

PROPOSITION 3. *The probability 1/v over the set of consecutive integer-valued bids: [0, 1,..., v-1] is a mixed-strategy Nash equilibrium. Further, in equilibrium expected profits are 1/2.*[7]

4.2. *Nash Equilibrium of the Asymmetric All-Pay Auction*

The calculation of the asymmetric all-pay auction with integer-valued bids is now discussed. The equilibrium expected payoffs S_i are given in equations (14) and (15). The density, $f_1(p_i)$, denotes bidder's 1 equilibrium probability that a price selected is p_i, where $f_1(p_i) \geq 0$ for $p_i = p_1, ..., v_2$.

$$S_1 = \left[\sum_{p_i=p_1}^{p_{k-1}} f_2(p_i)\right](v_1 - p_k) + f_2(p_k)(\frac{v_1}{2} - p_k) \quad (14)$$
$$+ \left[1 - \sum_{p_i=p_1}^{p_{k-1}} f_2(p_i)\right](-p_k)$$

Similarly:

$$S_2 = \left[\sum_{p_i=p_1}^{p_{k-1}} f_1(p_i)\right](v_2 - p_k) + f_1(p_k)(\frac{v_2}{2} - p_k) \quad (15)$$
$$+ \left[1 - \sum_{p_i=p_1}^{p_{k-1}} f_1(p_i)\right](-p_k)$$

where $p_k = p_1, ..., v_2$. Equations (14) and (15) can also be expressed as:

$$S_1 = \left[\sum_{p_i=p_1}^{p_{k-1}} f_2(p_i) + \frac{f_2(p_k)}{2}\right](v_1 - p_k) \quad (16)$$

[7] López-Acevedo (1995) examines the mixed strategy equilibria with continuous bid choices in this type of games.

$$+\left[1-\left[\sum_{p_i=p_1}^{p_{k-1}} f_2(p_i)+\frac{f_2(p_k)}{2}\right]\right](-p_k),$$

$$S_2=\left[\sum_{p_i=p_1}^{p_{k-1}} f_1(p_i)+\frac{f_1(p_k)}{2}\right](v_2-p_k) \tag{17}$$

$$+\left[1-\left[\sum_{p_i=p_1}^{p_{k-1}} f_1(p_i)+\frac{f_1(p_k)}{2}\right]\right](-p_k)$$

The $G_i(p_k)$ in equations (18) and (19) are modified "distribution functions" that allow for the event of ties:

$$G_1(p_k)=\sum_{p_i=p_1}^{p_{k-1}} f_1(p_i)+\frac{f_1(p_k)}{2}=\frac{p_k+S_2}{v_2}, \tag{18}$$

$$G_2(p_k)=\sum_{p_i=p_1}^{p_{k-1}} f_1(p_i)+\frac{f_1(p_k)}{2}=\frac{p_k+S_1}{v_1}, \tag{19}$$

where the final equality follows from (14) and (15). In order to obtain the support of the equilibrium mixed-strategy Nash equilibrium, consider a set of consecutive integer-valued bids: $[p_1, p_2, ..., v_2]$, where v_2 is the largest integer bid. As before, define p_L and p_H as the lowest and highest bids respectively that are selected with strictly positive probability, where $p_1 \leq p_L < p_H \leq v_2$. By evaluating (18) and (19) at p_H and using the fact that the sum of the densities up to (p_H) equals one, one obtains:

$$S_1=[1-f_2(p_H)]\,(v_1-p_H)+f_2(p_H)(\frac{v_1}{2}-p_H) \tag{20}$$

$$=v_1-p_H-f_2(p_H)\frac{v_1}{2}$$

Since $f_1(p_H)>0$ and $f_2(p_H)>0$, it follows from (20) and (21) that $S_1<v_1-p_H$ and:

$$S_2=[1-f_1(p_H)]\,(v_2-p_H)+f_1(p_H)(\frac{v_2}{2}-p_H) \tag{21}$$

$$=v_2-p_H-f_1(p_H)\frac{v_2}{2}$$

$S_2 < v_2 - p_H$. Now, we calculate the mixed strategy equilibrium for this model. Conjecture that $f_2(p_k) = 1/v_1$ and $f_1(p_k) = 1/v_2$ with the upper bound $p_H = v_2 - 1$ and the lower bound $p_1 = p_L = 0$, is a Nash equilibrium in mixed-strategies. Next, we verify that a seller is indifferent between the bids $0, 1, ..., v_2 - 1$. By evaluating (16) and (17) at $p_H = v_2 - 1$, one obtains:

$$S_1 = \left[F_2(v_2 - 2) + \frac{f_2(v_2 - 1)}{2}\right] v_1 - (v_2 - 1) \tag{22}$$

$$= \left[\frac{v_1 - v_2 + v_2 - 1}{v_1} + \frac{1}{2v_1}\right] v_1 - (v_2 - 1)$$

$$S_2 = \left[F_1(v_2 - 2) + \frac{f_1(v_2 - 1)}{2}\right] v_2 - (v_2 - 1) \tag{23}$$

$$= \left[\frac{v_2 - 1}{v_2} + \frac{1}{2v_2}\right] v_2 - (v_2 - 1)$$

It is straightforward to verify from (22) and (23) that $S_1 = v_1 - v_2 + 1/2$ and $S_2 = 1/2$. An analogous argument shows that at the lower bound, $p_L = 0$, $S_1 = v_1 - v_2 + 1/2$ and $S_2 = 1/2$, and similarly for intermediate prices. To summarize:

PROPOSITION 4. *In the Nash equilibrium of the asymmetric all-pay auction with discrete bid choices, firm 1 bids randomly with probability* $1/v_2$ *and firm 2 bids with probability* $1/v_1$ *in the support* $[0, ..., v_2]$. *Furthermore, expected profits are different in equilibrium.*

Next, the standard Nash equilibrium analysis of the all-pay auction is generalized to allow for endogenously determined decision "errors".

5. The Quantal Response Equilibrium[8]

The quantal or discrete response has its origins in stimulus/response models in biology and in statistical limited dependent variable models such as probabilistic choice (in economics and psychology). Probabilistic theories of choice can be divided in two basic types: constant utility models and random utility models. In the first interpretation the utility is constant but the decision rule is random (Luce, 1959; Tversky, 1972a). By contrast, the second interpretation assumes that

[8] This section draws heavily on López-Acevedo (1997).

utility is random while the decision rule is constant (Thurstone, 1927; McFadden, 1984). These approaches are formulated for individual decisions where the probability of making a decision is a function of the expected payoffs of all possible decisions.

There are many ways to model decision errors. One particularly simple approach is based on the discrete choice theory first proposed by Luce (1959). Let u_1 and u_2 denote the expected utility associated with decisions 1 and 2 respectively. Luce proposed a model in which choice probabilities are determined by ratios of expected utilities:

$$\Pr(choose \quad decision\ i) = \frac{u_i}{u_1 + u_2} \quad i = 1, 2 \tag{24}$$

These choice probabilities reflect boundedly rational behavior because the player does not always choose the decision with the highest utility.

In McFadden's approach (1984), the modeler can only imperfectly observe the characteristics influencing an individual's choice. For example, the u_i in the equation above represents the observed parts of an individual's utility, but the optimal decision may also depend on unobserved utility elements that are random from the point of view of an outside observer. The distribution of the random utility elements determines the form of the probabilistic choice function (e.g. logit, probit), as discussed below. These choice functions are also called "quantal response functions".

The quantal response functions discussed above are used to model individual decisions. Capturing decision error in a way that is clearly spelled out and not ad hoc is a difficult task. The quantal response equilibrium does this based on elements borrowed from the discrete quantal choice theory developed by Luce (1959), McFadden (1984), and Thurstone (1927). The added complexity of applying the quantal response equilibrium to game theory —in contrast to individual choice— is that the choice probabilities of the players have an important interactive component, since they are simultaneously determined in equilibrium. In a quantal response equilibrium, a player's beliefs about others' actions will determine the player's own expected payoffs, which in turn determine the player's choice probabilities via a quantal response function. The model is closed by requiring the choice probabilities to be consistent with the initial beliefs.[9]

[9] Other authors have provided alternative explanations such as that each player privately observes his/her payoff with some noise and the distribution over the noise is common knowledge. Players do not update given their realization (i.e. they are not Bayes consistent).

To illustrate the effects of decision errors in a market model, consider the quantal response equilibrium for a simple Prisoner's Dilemma game. In this game, each player simultaneously chooses between cooperation C, and non-cooperation N. The profits from defection, π_d, exceed those from cooperation, π_c, which in turn exceed the profit π_n from the Nash equilibrium: $\pi_d > \pi_c > \pi_n > 0$. The only Nash equilibrium outcome is (π_n, π_n).

		Player 2: C	Player 2: N
Player 1	C	π_c π_c	0 π_d
	N	π_d 0	π_n π_n

Next, consider the effects of decision errors determined in (24). Let σ denote the probability that player 2 chooses the cooperative decision C. Given this probability, player 1's expected payoff is $u_C = \sigma\pi_c$ for decision C and $u_N = \sigma\pi_d + (1-\sigma)\pi_n$ for decision N. Using Luce's choice function (24), player 1 will choose decision C with probability:

$$\Pr(\textit{choose} \quad \textit{decision } C) = \frac{\sigma\pi_c}{\sigma\pi_c + \sigma\pi_d + (1-\sigma)\pi_n} \tag{25}$$

The equilibrium consistency requirement is that choice probabilities correspond to beliefs. In particular, the right side of equation (25) must equal σ, which provides an equation that can be solved:[10]

$$\sigma = \frac{\pi_c - \pi_n}{\pi_c - \pi_n + \pi_d} \tag{26}$$

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[10] Note that the Nash equilibrium condition, $\sigma = 0$, does not satisfy (26).

since N is a dominant strategy in the Nash game without errors, as long as $\pi_d > \pi_c, \sigma$ can be interpreted as the probability of making an error.

To illustrate the random utility interpretation, consider the following example. Let the subject's utility derived from alternatives 1 and 2 be written as:

$$V_1^* = v_1 + \varepsilon_1$$

$$V_2^* = v_2 + \varepsilon_2$$

where ε_1 are the residual random elements which are *i.i.d.* log Weibull distributed[11] with parameter λ. Now consider the probability that the first alternative will be chosen:

$$Pr(\text{choose } 1) = Pr(v_1 + \varepsilon_1 > v_2 + \varepsilon_2)$$

$$= Pr(v_1 - v_2 > \varepsilon_2 - \varepsilon_1)$$

$$= F(v_1 - v_2),$$

where $F(\cdot)$ denotes a cumulative distribution. Then the probability that an individual chooses alternative 1 can be expressed in terms of the logistic error function (McFadden, 1984):

$$Pr(1) = \frac{e^{\lambda v_1}}{e^{\lambda v_2} + e^{\lambda v_1}} \tag{27}$$

As $1/\lambda$ goes to ∞ in equation (27), it can be shown that the variance of the error terms tends to infinity. Thus the individual will choose between decisions 1 and 2 with equal probability, regardless of the expected payoffs, as can be seen from the limiting case of (27) with $\lambda = 0$. The error variance, $1/\lambda$ goes to 0 as λ goes to ∞, and therefore, it follows from (27) that the probability of choosing the option with the higher expected payoff goes to 1.

The distribution of the error term determines the probabilistic choice function (e.g. logit, probit and power-function). These random residuals can be interpreted as being caused by decision errors.

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[11] The log-Weibull distribution is as follows: $F(x) = exp\left[- exp\left(\frac{A-x}{B}\right)\right]$, where A is a location parameter and B is a scale parameter

Under the decision error interpretation, these choice probabilities reflect boundedly rational behavior, in the sense that an individual does not always choose the decision with the highest utility. These choice probabilities reflect a tendency toward utility maximization because a non-optimal choice is less likely when the difference in the underlying utilities is large. The quantal response equilibrium incorporates the framework into an equilibrium analysis.

Following McKelvey and Palfrey (1994) a quantal response equilibrium is a fixed point in choice probabilities. Define π as the set of all possible combinations of the expected payoffs for all players in a finite normal form game. Let δ be the Cartesian product of the mixed strategies for all players, and let $\hat{p}$ be an element of δ, i.e. $\hat{p}$ specifies a particular mixed strategy for each player. Denote a vector of all expected payoffs as $e(\hat{p})$. Thus, $e(\hat{p})$ maps a particular array of mixed strategies, $\hat{p}$, into a vector of players' expected payoffs, π. A discrete choice function σ maps expected payoffs into a mixed strategy for a single player. The function σ is assumed to be continuous and monotonically increasing in the payoffs. Let T^{σ} represent the resulting mapping from the set of all possible combinations of players' expected payoffs to their choice probabilities, $T^{\sigma} : \pi \rightarrow \delta$.

To summarize, $e(\hat{p}) : \delta \rightarrow \pi$ maps mixed strategy probabilities to expected payoffs, and $T^{\sigma} : \pi \rightarrow \delta$ maps expected payoffs to mixed strategy probabilities.

The equilibrium is a fixed point:

DEFINITION. *A Quantal Response Equilibrium (QRE) is a $\hat{p}$ such that $\hat{p} = T^{\sigma}(e(\hat{p}))$.*

The Brouwer fixed point theorem implies the existence of such an equilibrium, since $T^{\sigma}(e(\hat{p}))$ is a continuous function that maps a compact set δ onto itself.[12]

In what follows, two common specifications of the quantal response equilibrium are examined: *the additive-error quantal response equilibrium and the multiplicative-error quantal response equilibrium.* The first functional from is based on random utility maximization with additive error terms and it is a standard approach to model decision errors in normal-form games (McKelvey and Palfrey, 1994). The second specification follows from random utility maximization with multiplicative error terms. This framework has proved to be a useful way to model decision errors in models of price competition since it often leads to tractable solutions and comparative statistics results (López-Acevedo, 1997).

[12] This result is due to McKelvey and Palfrey (1994).

Our primary focus in section 6 and section 7 is to derive a general statement of the Nash and quantal response equilibrium equivalence of the all-pay auction. However, in the process of finding it, propositions that are needed to derive this equivalence are established.

6. Equivalence of Equilibria with Continuous Bid Choices

This section provides an analysis of the quantal response equilibrium with additive error terms for the symmetric and asymmetric all-pay auction with continuous bid choices (a similar proof for the multiplicative-error quantal response equilibrium is derived in the appendix).

The following proposition presents the conditions under which equivalence between the Nash and the quantal response equilibria occurs in the symmetric all-pay auction model.

6.1. *Quantal Response Equilibrium of the Symmetric All-Pay Auction*

PROPOSITION 5. *Consider a game in which the strategy space, S_i, of player i is an interval of actions, $s \in [0, v]$. Suppose there exists a Nash equilibrium to the game in which player i plays each action with probability $f(s) = 1/v$. Let $\pi_i^*(s)$ denote i's profit given that all other players play their equilibrium mixed strategies, and suppose furthermore that $\pi_i^*(s) \geq 0 \; \forall \; s \in [0, v]$. Then there exists an additive-error quantal response equilibrium when the error structure satisfies $\varepsilon_i \geq 0(F(0) = 0$, no mass points at zero) and the errors are identically independently distributed. Furthermore, this additive-error quantal response equilibrium is identical to the Nash equilibrium described above.*

PROOF. It suffices to show that bidder i will choose strategy s *with equal probability*, given that the other bidders choose strategies as described in the proposition. Since $\pi_i^*(s) = \bar{\pi} > 0 \; \forall \; s \in [0, v]$, it needs to be shown that:

$$Pr(s) = \Pr(\bar{\pi} + \varepsilon_j = \max_{j \text{ is } \max} \bar{\pi} + \varepsilon_i) = \frac{1}{n}. \tag{28}$$

Let F be the common cumulative distribution function of the ε_1, and let f be the corresponding density. Equation (28) is true since:

$$Pr(\bar{\pi} + \varepsilon_j = \max_{j \text{ is } \max} \bar{\pi} + \varepsilon_i) = \int_0^\infty f(\varepsilon) \prod_{i \neq j} F(\varepsilon) d\varepsilon$$

$$= \int_0^\infty (F(\varepsilon))^{n-1} f(\varepsilon) d\varepsilon = \frac{(F(\varepsilon))^n}{n} \Big|_0^\infty = \frac{1}{n} \tag{29}$$

Notice that in equation (29), the i subscript is dropped from the error terms in the second expression since it is assumed that the errors are identically and independently distributed. The appendix shows the results assuming identically, independently distributed errors but in the case of multiplicative errors. The intuition behind this result is that expected profits in the mixed-strategy Nash equilibrium are equal for all bids in the support $[0, v]$. Hence if the rival is using his Nash equilibrium, the bidder's best quantal response is to spread bid decisions uniformly in the support $[0, v]$. ∎

6.2. *Quantal Response Equilibrium of the Asymmetric All-Pay Auction*

The next proposition examines the quantal response equilibrium of the asymmetric model.

PROPOSITION 6. *Consider a game in which the strategy space, S_i, of player i is an interval of actions, $S \in [0, v_2], i = 1, 2$. Suppose there exists a Nash equilibrium to the game at which bidder 1 plays each action with probability $f_1(s) = 1/v_2$ and bidder 2 with probability $f_2(s) = 1/v_1$. Let $\pi_1^*(s)$ denote i's profit given that the other bidder plays his equilibrium mixed strategy, and suppose furthermore that $\pi_i^*(s) \geq 0 \ \forall \ s \in [0, v_2]$. Then there exist an additive-error quantal response equilibrium when the error structure satisfies $\varepsilon_i \geq 0 (F(0) = 0$, no mass points at zero) and the errors are identically and independently distributed. Furthermore, this additive-error quantal response equilibrium is identical to the Nash equilibrium described above.*

PROOF. It suffices to show that player 1 will choose strategy s with equal probability, $1/v_2$, given that player 2 chooses his strategies as described in the proposition. Since $\pi_i^*(s) = \bar{\pi}_i \geq 0 \ \forall \ s \in [0, v_2]$, it needs to be shown that:

$$Pr(\bar{\pi}_1 + \varepsilon_{1j} = \max_{j \text{ is } \max} \bar{\pi}_1 + \varepsilon_{1i}) = \frac{1}{n_1}, \tag{30}$$

$$Pr(\bar{\pi}_2 + \varepsilon_{2j} = \max_{j \ is \ max} \bar{\pi}_2 + \varepsilon_{2i}) = \frac{1}{n_2}.$$

This is true since:

$$Pr(\bar{\pi}_1 + \varepsilon_{1j} = \max_{j \ is \ max} \bar{\pi}_1 + \varepsilon_{1i}) = \int_0^\infty f(\varepsilon_1) \prod_{j \neq 1} F(\varepsilon_1) d\varepsilon_1 \qquad (31)$$

$$= \frac{(F(\varepsilon_1))^{n_1}}{n_1} \Big|_0^\infty = \frac{1}{n_1},$$

$$Pr(\bar{\pi}_2 + \varepsilon_{2j} = \max_{j \ is \ max} \bar{\pi}_2 + \varepsilon_{2i}) = \int_0^\infty f(\varepsilon_2) \prod_{j \neq 1} F(\varepsilon_2) d\varepsilon_2$$

$$= \frac{(F(\varepsilon_2))^{n_2}}{n_2} \Big|_0^\infty = \frac{1}{n_2},$$

The equivalence between the Nash and quantal response equilibria of the asymmetric model arises because a player's i expected profits in the mixed-strategy Nash equilibrium are equal at all bids in the support $[0, v_2]$. Hence, if player 2 is using his Nash equilibrium, player 1's best quantal response is to spread bid decisions uniformly with probability $1/v_2$ in the support $[0, v_2]$. Similarly, if player 1 is using his Nash equilibrium, player 2's best quantal response is to spread bid decisions uniformly with probability $1/v_1$ in the support $[0, v_2]$. ∎

7. Equivalence of Equilibria with Discrete Bid Choices

Thus far, it has been shown that the Nash and quantal response equilibria for continuous bid choices are identical. This section is devoted to examining such results in the discrete choice framework. As before, the symmetric case is considered first. Then, the asymmetric model is examined. The intuition gained from the previous section can be useful in deriving the quantal response equilibrium for discrete choices.

7.1. *Equivalence of Symmetric Equilibria*

PROPOSITION 7. *Consider a game in which the strategy space, S_{im}, of player i is a discrete set of actions, $s_{im} \in S_{im}, m = 1...n$, where*

$s_{im} < v$. Suppose there exists a Nash equilibrium to the game, s^*_{im}, at which player i plays each action in $\hat{S}_{im} \subset S_{im}$, with equal probability (i.e. the Nash equilibrium is $s^*_{im} = 1/k \ \forall \ s^*_{im} \in \hat{S}_{im}$, where $k =$ number of elements in $\hat{S}_{im}$). Let $\pi^*_i(s_{im})$ denote i's profit given that all other bidders play their equilibrium mixed strategies, and suppose furthermore that $\pi^*_i(s_{im}) > 0 \ \forall \ s_{im} \in S_{im}$ while $\pi^*_i \leq 0 \ \forall \ s_{im} \notin S_{im}$. Then there exists an additive-error quantal response equilibrium when the error structure satisfies $\varepsilon_{im} = 0(F(0) = 0$, no mass points at zero) and the errors are identically and independently distributed. Furthermore, this additive-error quantal response equilibrium is identical to the Nash equilibrium described above.

PROOF. It suffices to show that player i will choose each strategy in $\hat{S}_{im}$ with equal probability, given that the other bidders choose strategies as described in the proposition. Clearly, since $u_{i1} = \pi^*_i(s_{il}) + \varepsilon_{il}$, no strategy outside $\hat{S}_{im}$ will be chosen (to do so would yield non-positive payoffs for all realizations of s_{im}, but non-negative payoffs are guaranteed for $s_{im} \in \hat{S}_{im}$, and the probability of "ties" at zero is zero since $F(0) = 0$). Finally, since $\pi^*_i(s_{ij}) = \pi^*_i(s_{il}) = \bar{\pi} > 0 \ \forall \ s_{ij}, s_{il} \in S_{im}$, it remains to be shown that:

$$Pr(\bar{\pi} + \varepsilon_{ij} = \max_{j \ is \ \max} \bar{\pi} + \varepsilon_{i1}) = \frac{1}{k_i} \tag{32}$$

This is true since:

$$Pr(\bar{\pi} + \varepsilon_{ij} = \max_{j \ is \ \max} \bar{\pi} + \varepsilon_{i1}) = \int_0^\infty f(\varepsilon) \prod_{j \neq 1} F(\varepsilon) d\varepsilon \tag{33}$$

$$= \frac{(F(\varepsilon))^{k_i}}{k_i} \Big|_0^\infty = \frac{1}{k_i},$$

Notice that in equation (33), the i subscript is dropped from the error terms in the second expression since it is assumed that the errors are identically and independently distributed. ∎

7.2. *Equivalence of Asymmetric Equilibria*

Next the quantal response equilibria of the asymmetric all-pay auction with discrete bid choices are examined.

PROPOSITION 8. *Consider a game in which the strategy space, S_{im}, of the player i is a discrete set of actions, $s_{im} \in S_{im}, m = 1...n$ and $i = 1,2$ where $s_{im} < v_2$. Suppose there exists a Nash equilibrium to the game, s^*_{im}, at which player i plays each action in $\hat{S}_{im} \subset S_{im}$, with equal probability (i.e. the Nash equilibrium, $s^*_{1m} = 1/k_2$ and $s^*_{2m} = 1/k_1 \; \forall \; s^*_{im} \in S_{im}$, where k_2 = number of elements in S_{im} and k_1 = number of elements in the strategy space A_{im} with $\hat{S}_{im} \subset A_{im}$). Let $\pi^*_i(s_{im})$ denote i's profit given that the other player plays his equilibrium mixed strategies, and suppose furthermore that $\pi^*_i(s_{im}) > 0 \; \forall \; s_{im} \in S_{im}$ while $\pi^*_i \leq 0 \; \forall \; s_{im} \notin S_{im}$. Then there exists an additive-error quantal response equilibrium when the error structure satisfies $\varepsilon_{im} = 0(F(0) = 0$, no mass points at zero) and the errors are identically and independently distributed. Furthermore, this additive-error quantal response equilibrium is identical to the asymmetric Nash equilibrium described above.*

PROOF. It suffices to show that player i will choose each strategy in S_{im} with equal probability given that the other player chooses strategies as described in the proposition. Clearly, since $u_{il} = \pi^*_i(s_{il}) + \varepsilon_{il}$, no strategy outside S_{im} will be chosen (to do so would yield non-positive payoff for all realizations of s_{im}, but non-negative payoffs are guaranteed for $s_{im} \in S_{im}$, and the probability of "ties" at zero is zero since $F(0) = 0$). Finally, since $\pi^*_i(s_{ij}) = \pi^*_i(s_{il}) = \bar{\pi} > 0 \; \forall \; s_{ij}, s_{il} \in S_{im}$, it remains to be shown that:

$$Pr(\bar{\pi}_1 + \varepsilon_{1j} = \max_{j \; is \; \max} \bar{\pi}_1 + \varepsilon_{11}) = \frac{1}{k_1} \tag{34}$$

$$Pr(\bar{\pi}_2 + \varepsilon_{2j} = \max_{j \; is \; \max} \bar{\pi}_2 + \varepsilon_{21}) = \frac{1}{k_2}$$

This is true since:

$$Pr(\bar{\pi}_1 + \varepsilon_{1j} = \max_{j \; is \; \max} \bar{\pi}_1 + \varepsilon_{11}) = \int_0^{\infty} f(\varepsilon_1) \prod_{j \neq 1} F(\varepsilon_1) d\varepsilon_1 \tag{35}$$

$$= \frac{(F(\varepsilon_1))^{k_1}}{k_1} \Big|_0^{\infty} = \frac{1}{k_1}$$

$$Pr(\bar{\pi}_2 + \varepsilon_{2j} = \max_{j \; is \; \max} \bar{\pi}_2 + \varepsilon_{21}) = \int_0^{\infty} f(\varepsilon_2) \prod_{j \neq 1} F(\varepsilon_2) d\varepsilon_2$$

$$= \frac{(F(\varepsilon_2))^{k_2}}{k_2} \Big|_0^\infty = \frac{1}{k_2}$$

The above set of propositions show why the quantal response equilibrium and Nash distribution will be the same when the Nash mixed distribution is a uniform distribution and the error terms are identically and independently distributed. Next, the quantal response equilibrium of the all-pay auction for two particular parametrizations is computed. ∎

8. Parameterizations of Quantal Response Equilibrium

In what follows, the logit and the power-function equilibria of the all-pay auction are computed. The first one is based on random utility maximization with additive error terms. The second equilibrium specification follows from random utility maximization with multiplicative error terms. It is well known that if the error terms are log Weibull distributed then the best response functions take a logistics form. Thus for any $\lambda > 0$, the logistic quantal response equilibrium condition is given by:

$$f(p) = \frac{e^{\lambda[vF(p)-p]}}{\mu} \tag{36}$$

$$\mu = \int_0^v e^{\lambda[vF(x)-x]} dx$$

where μ is a constant independent of p. The first equation in (36) is a nonlinear differential equation in the price distribution $F(p)$. In order to obtain $F(p)$ we first multiply both sides of the top equation in (36) by $-\lambda v e^{-\lambda v F(p)}$ which yields

$$-\lambda v f(p) e^{-\lambda v F(p)} = \frac{-\lambda v e^{-\lambda p}}{\mu} \tag{37}$$

Integrating over all values of p, i.e. from p_a to p^*, we have

$$\int_{p_a}^{p*} -\lambda v f(p) e^{-\lambda v F(p)} dp = \int_{p_a}^{p*} \frac{-\lambda v e^{-\lambda p}}{\mu} dp \tag{38}$$

The resulting equation is written as

$$e^{-\lambda v F(\bar{p})} \Big|_{F(p_a)}^{F(p*)} = \frac{ve^{-\lambda p}}{\mu} \Big|_{p_a}^{p*} \tag{39}$$

The next task is to determine μ from an analysis of boundary conditions. Since negative prices produce no profits, conjecture that $F(p_a) = 0$, with $p_a = 0$. Let $\bar{p}$ denote the upper bound of the bid distribution. Using the boundary condition, $F(0) = 0$, equation (39) becomes:

$$e^{-\lambda v F(\bar{p})} - 1 = \frac{v}{\mu}[e^{-\lambda \bar{p}} - 1] \tag{40}$$

Now consider the upper bound $\bar{p}$. From (37), $\bar{p} > v$ implies that $f(p) > 0$. Since bids above v produce 0 profits, conjecture that $F(\bar{p}) = 1$, with $\bar{p} = v$. This conjecture in turn implies that $\mu = v$. Substituting μ back into (40), we have

$$e^{-\lambda v F(p)} = e^{-\lambda p} \tag{41}$$

It is readily verified from (41) that the equilibrium probability functions is:

$$F(p) = \frac{p}{v} \tag{42}$$

which also satisfies the working assumption used above: $F(0) = 0$ and $F(v) = 1$. Equation (42) is also the Nash equilibrium of the symmetric all-pay auction model.

Next the power-function quantal response equilibria are computed. As shown in the appendix, when the error terms are distributed as equation (52) the relevant decision rule is the power function. The power-function quantal response equilibrium condition must satisfy:

$$f(p_k) = \frac{\left(\left[F(p_{k-1}) + \frac{f(p_k)}{2}\right] v - p_k\right)^{\lambda}}{\mu} \tag{43}$$

$$\mu = \sum_{x=0}^{v} \left(\left[F(x-1) + \frac{f(x)}{2}\right] v - x\right)^{\lambda}$$

where μ is a constant independent of prices. The probability density in (43) is obtained by solving recursively the first equation in (43), beginning with the lowest bid and working upward. For simplicity, let $\lambda = 1$. Since ties are possible with integer-valued bids, it follows that $f(0)/2 > 0$. By evaluating (43) at the lowest bid $p_k = 0$, it follows that $\mu = v/2$. The substitution of μ back into (43) results in the following expression:

$$F(p_{k-1})v = p_k \tag{44}$$

Let $p_k = z$. Then, the above equation is written as:

$$\sum_{i=0}^{z-1} f(i)v = z \tag{45}$$

for $z = 1, ..., v$. Consider $z = 1$. Then, it follows from (42) that $f(0) = 1/v$. Assume $z = 2$. Then, equation (45) yields:

$$f(0) + f(1) = \frac{2}{v} \tag{46}$$

Since $f(0) = 1/v$, it must be the case that $f(1) = 1/v$ in (46). A similar argument shows that $z = v$ yields:

$$\frac{v-1}{v} + f(v-1) = 1 \tag{47}$$

It is readily verified from (47) that $f(v-1) = 1/v$. Now consider $\lambda > 1$. Notice that $f(0) > 0$ implies that $\mu = (f(0))^{-1}(v/2)$. Conjecture that the equilibrium probability is $1/v$. From (42), it follows that $\mu = v(1/2)$. By evaluating (47) at $p_k = v - 1$ and using the solution for μ, one obtains:

$$f(v-1) = \frac{2^{\lambda}\left(\left[F(v-2) + \frac{f(v-1)}{2}\right]v - (v-1)\right)^{\lambda}}{v} \tag{48}$$

Substituting the conjecture $1/v$ in both sides of (48), it follows that $1/v$ satisfies (47). A similar argument shows that the uniform distribution, $1/v$, satisfies intermediate bid values.

Thus, for any functional form of the error term, the Nash and quantal response equilibria of the all-pay auction are the same as long as the error terms are identically and independently distributed.

9. Conclusions

This paper examined the quantal response equilibrium of the (first-price) all-pay auction model. A striking result derived is that for any structure of the error term the Nash and quantal response equilibrium of the all-pay auction are identical if the error terms are independently and identically distributed. This is because the expected profits in the mixed?strategy Nash equilibrium are equal at all bids in the support of the equilibrium distribution. Hence, if the rival is using his Nash equilibrium, the seller's best response is to spread bid decisions uniformly in the support. In addition, it is shown that this result holds in the asymmetric case with two players. This paper also presents a step-by-step procedure for calculating the quantal response equilibria, which was summarized in section 8. This procedure was applied to two parametric quantal response functions, which we call the power function and the logit. We also showed how to calculate mixed-strategy Nash equilibria using the discrete bid choices that are common in laboratory experiments.

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Appendix

In this appendix I prove the equivalence of the Nash and quantal response equilibria with multiplicative error terms of the all-pay auction for continuous and discrete bid choices.

PROPOSITION 9. *Consider a game in which the strategy space, S_i, of player i is an interval of actions, $s \in [0, v]$. Suppose there exists a Nash equilibrium to the game at which player i plays each action with probability $f(s) = 1/v$. Let $\pi_i^*(s)$ denote i's profit given that all other players play their equilibrium mixed strategies, and suppose furthermore that $\pi_i^*(s) \geq 0 \ \forall \ s \in [0, v]$. Then there exists a multiplicative-error quantal response equilibrium when the error structure satisfies $\varepsilon_i \geq 0(F(0) = 0$, no mass points at zero) and the errors are identically and independently distributed. Furthermore, this multiplicative-error quantal response equilibrium is identical to the Nash equilibrium described above.*

PROOF. It suffices to show that player i will choose strategy s with equal probability, given that the other players choose strategies as described in the proposition. Since $\pi_i^*(s) = \bar{\pi}0 \ \forall \ s \in [0, v]$, it needs to be shown that:

$$Pr(\bar{\pi}\varepsilon_j = \max_{j \ is \ \max} \bar{\pi}\varepsilon_i) = \frac{1}{n} \tag{49}$$

Making a logarithmic transformation, we have:

$$Pr(\ln \bar{\pi} + \ln \varepsilon_j = \max_{j \ is \ \max} \ln \bar{\pi} + \ln \varepsilon_i) \tag{50}$$

Let $\bar{v} = \ln \bar{\pi}$. Then, equation (50) can be written as:

$$Pr(\bar{v} + \ln \varepsilon_j = \max_{j \ is \ \max} \bar{v} + \ln \varepsilon_i) \tag{51}$$

Let $G(^*)$ denote the distribution of ε such that:[13]

$$G(\varepsilon) = e^{-e(\varepsilon)^{-\lambda}}, \qquad \varepsilon \in [0 \ , \infty) \qquad \lambda > 0 \tag{52}$$

Define a transformation of the error term: $\kappa = \ln$ or $\varepsilon = e^{\kappa}$. Substitute e^{κ} for ε in (52) to obtain the distribution function:

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[13] In the analysis that follows, the i subscript is dropped from the error terms since the errors are *i.i.d.*

$$H(k) = e^{-e^{-\lambda\kappa}} \tag{53}$$

which is a log Weibull distribution with parameter λ. When an additive random utility error, κ, is log Weibull distributed, Luce and Suppes (1967) have shown that the standard decision rule is the logit:

$$\frac{e^{\lambda\bar{v}}}{\sum_i^n e^{\lambda\bar{v}}} \tag{54}$$

Since the logarithmic transformation of the multiplicative error in (49) is additive in the logarithm $\bar{\pi}$, the relevant probabilistic choice function is the logit formulation with $\bar{v}$ replaced by log $\bar{\pi}$ as follows:

$$e^{\lambda\overline{v}} = e^{\lambda \log \overline{\pi}} = \overline{\pi}^{\lambda} \tag{55}$$

and the logistic choice rule in equation (53) reduces to the power function rules in (56). The power-function quantal response equilibrium selects all options with equal $\bar{\pi} \geq 0$:

$$\frac{\bar{\pi}^{\lambda}}{n\bar{\pi}^{\lambda}} = \frac{1}{n} \tag{56}$$

even when $\pi = 0$, since $lim\ \pi \to 0^+$. ∎

Multiplicative-Error Quantal Response Equilibrium of the Symmetric All-Pay Auction

PROPOSITION 10. *Consider a game in which the strategy space, S_{im}, of player i is a discrete set of actions, $s_{im} \in S_{im}, m = 1...n$. Suppose there exists a Nash equilibrium, s^*_{im}, to the game at which player i plays each action in $\hat{S}_{im} \subset S_{im}$ with equal probability (i.e. $s^*_{im} = 1/k\ \forall\ s^*_{im} \in \hat{S}_{im}$, where k = number of elements in $\hat{S}_{im}$). Let $\pi^*_i(s_{im})$ denote i's profit given that all other players play their equilibrium mixed strategies, and suppose furthermore that $\pi^*_i(s_{im}) > 0\ \forall\ s_{im} \in \hat{S}_{im}$ while $\pi^*_i \leq 0\ \forall\ s_{im} \notin \hat{S}_{im}$. Then there exists a multiplicative-error quantal response equilibrium when the error structure satisfies $\varepsilon_{im} \geq 0 (F(0) =$ no mass points at zero) and the errors are identically and independently distributed. Furthermore, this multiplicative-error quantal response equilibrium is identical to the Nash equilibrium described above.*

PROOF. It suffices to show that player i will choose each strategy in $\hat{S}_{im}$ with equal probability, given that the other players choose strategies as described in the proposition. Clearly, since $u_{il} = \pi_i^*(s_{il})\varepsilon_{il}$, no strategy outside $\hat{S}_{im}$ will be chosen (to do so would yield non-positive payoff for all realizations of s_{im}, but non-negative payoffs are guaranteed for $s_{im} \in \hat{S}_{im}$, and the probability of "ties" at zero is zero since $F(0) = 0$). Finally, since $\pi_i^*(s_{ij}) = \pi_i^*(s_{il}) = \bar{\pi} > 0 \; \forall \; s_{ij}, s_{il} \in \hat{S}_{im}$, it remains to be shown that:

$$Pr(\bar{\pi}\varepsilon_{ij} = \max_{j \; is \; \max} \bar{\pi}\varepsilon_{i1}) = \frac{1}{k} \tag{57}$$

This is true since:

$$Pr(\bar{\pi}\varepsilon_{ij} = \max_{j \; is \; \max} \bar{\pi}\varepsilon_{i1}) = \int_0^\infty f(\varepsilon) \prod_{j \neq 1} F(\varepsilon) d\varepsilon \tag{58}$$

$$= \frac{(F(\varepsilon))^k}{k} \Big|_0^\infty = \frac{1}{k}$$

∎

Multiplicative-Error Quantal Response Equilibrium of the Asymmetric All-Pay Auction

PROPOSITION 11. *Consider a game in which the strategy space, S_{im}, of player i is a discrete set of actions, $s_{im} \in S_{im}, m = 1, \ldots, n$ and $i = 1, 2$, where $s_{im} < v_2$. Suppose there exists a Nash equilibrium to the game, s_{im}^*, at which player i plays each action in $\hat{S}_{im} \subset S_{im}$, with equal probability (i.e. the Nash equilibrium, $s_{1m}^* = 1/k_2$ and $s_{2m}^* = 1/k_1 \; \forall \; s_{im}^* \in \hat{S}_{im}$, where k_2 = number of elements in $\hat{S}_{im}$ and k_1 = number of elements in the strategy space $A_{im}, \hat{S}_{im} \subset A_{im}$). Let $\pi_i^*(s_{im})$ denote i's profit given that all other players play their equilibrium mixed strategies, and suppose furthermore that $\pi_i^*(s_{im}) > 0 \; \forall \; s_{im} \in \hat{S}_{im}$ while $\pi_i^* \leq 0 \; \forall \; s_{im} \notin \hat{S}_{im}$. Then there exists a multiplicative-error quantal response equilibrium when the error structure satisfies $\varepsilon_{im} \geq 0 (F(0) = 0$, no mass points at zero) and the errors are identically and independently distributed. Furthermore, this multiplicative-error quantal response equilibrium is identical to the Nash equilibrium described above.*

PROOF. It suffices to show that player i will choose each strategy in $\hat{S}_{im}$ with equal probability given that the other player chooses strategies as described in the proposition. Clearly, since $u_{il} = \pi_i^*(s_{il})\varepsilon_{il}$, no strategy outside $\hat{S}_{im}$ will be chosen (to do so would yield non-positive payoff for all realizations of s_{im}, but non-negative payoffs are guaranteed for $s_{im} \in \hat{S}_{im}$, and the probability of "ties" at zero is zero since $F(0) = 0$). Finally, since $\pi_i^*(s_{ij}) = \pi_i^*(s_{il}) = \bar{\pi} > 0 \; \forall \; s_{ij}, s_{il} \in S_{im}$, it remains to be shown that:

$$Pr(\bar{\pi}_1\varepsilon_{1j} = \max_{j \text{ is max}} \bar{\pi}_1\varepsilon_{11}) = \frac{1}{k_1} \tag{59}$$

$$Pr(\bar{\pi}_2\varepsilon_{2j} = \max_{j \text{ is max}} \bar{\pi}_2\varepsilon_{21}) = \frac{1}{k_2}.$$

This is true since:

$$Pr(\bar{\pi}_1\varepsilon_{1j} = \max_{j \text{ is max}} \bar{\pi}_1\varepsilon_{11}) = \int_0^\infty f(\varepsilon_1) \prod_{j \neq 1} F(\varepsilon_1) d\varepsilon_1 \tag{60}$$

$$= \frac{(F(\varepsilon_1))^{k_1}}{k_1} \Big|_0^\infty = \frac{1}{k_1}$$

$$Pr(\bar{\pi}_2\varepsilon_{2j} = \max_{j \text{ is max}} \bar{\pi}_2\varepsilon_{21}) = \int_0^\infty f(\varepsilon_2) \prod_{j \neq 1} F(\varepsilon_2) d\varepsilon_2$$

$$= \frac{(F(\varepsilon_2))^{k_2}}{k_2} \Big|_0^\infty = \frac{1}{k_2}.$$

∎