NOTES ON THE SUBOPTIMALITY RESULT OF J. D. GEANAKOPLOS AND H. M. POLEMARCHAKIS (1986)*

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Resumen: J. D. Geanakoplos and H. M. Polemarchakis (1986) demuestran la suboptimalidad restringida de asignaciones de equilibrio en economías de dos periodos con mercados incompletos. Ellos perturban precios de activos de equilibrio cuando los mercados son incompletos de grado uno. Dado que los precios no parametrizan la economía, no se puede obtener un resultado genérico de dicha forma. Estas notas ofrecen una versión detallada de su demostración en la que perturbamos utilidades y dotaciones.

Abstract: J. D. Geanakoplos and H. M. Polemarchakis (1986) prove the generic constrained suboptimality of equilibrium allocations in two period economies with incomplete markets. They perturb asset prices at equilibrium when the degree of market incompleteness equals one. Since prices do not parameterize the economy, a generic result cannot be obtained in such a way. These notes provide a detailed version of their proof in which utilities and endowments are perturbed.

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Palabras clave: mercados de activos incompletos, suboptimalidad restringida, teoría de transversalidad, incomplete asset markets, constrained suboptimality, transversality theory.

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1. Introduction

J. D. Geanakoplos and H. M. Polemarchakis (1986) henceforth, GP showed that when real assets are traded in two-period economies with more than a single good, and markets are incomplete, then the equilibrium allocation is *constrained suboptimal*, i.e., even if the "planner" is restricted to using only the existing assets to obtain the reallocation, he is able to induce an *improvement* over the equilibrium allocation. This result has become a cornerstone for subsequent research in the area; in particular, it sheds light onto the open question of analyzing the optimality of equilibrium allocations in pure exchange OLG economies with sequentially incomplete markets when price effects are allowed for.

The key feature of the proof by GP is to show that (i) with incomplete markets, the ratios of marginal utilities of income across states differ *generically* across agents, a result which they use to show that (ii) with more than a single commodity, a *price effect* can be induced in such a way as to cause a *welfare improvement*. To prove result (i) above, GP perturb asset prices at equilibrium when the degree of market *incompleteness* equals one. However, since prices are not *fundamentals* that parameterize the economy, a generic result cannot be obtained in such a way. Accordingly we provide, in section 5, an alternative proof of result (i) above which does not depend on the dimension of the market incompleteness and in which utilities and endowments are perturbed.

Also, the original proof by GP of result (ii) above, though correct and brilliant, skips many details in order to shorten the presentation. We believe that understanding the problem requires one to have the relevant details and, accordingly, we provide them and complete the arguments following the sketches given by GP. In this respect, our endeavor allows the reader to appreciate better the nature of the contribution of GP.

To prove that a welfare improvement is derived from a relative price effect, one must show that a property of linear independence is generically satisfied for a set of vectors derived from the *income effect* vectors.[1] To guarantee that this property holds, an upper bound needs to be imposed on the number of agents, as GP do, which in turn requires that the number of agents relative to the number of goods in the economy be sufficiently small. This is controversial since, from

[1] For each agent, an income effect vector reflects the changes in his demand for commodities as a consequence of changes in his income.

the competitive equilibrium perspective, one usually has in mind an economy where the number of agents is large relative to the number of commodities. Citanna, Kajii and Villanacci (1998) henceforth, CKV have proved the GP result without imposing an upper bound on the number of agents. However, their description of the *intervention* differs from the one used by GP in that (a) agents are allowed to retrade the assets allocated at the intervention, and (b) the planner makes *lump-sum* transfers in some goods. As we show, the result by CKV follows precisely because feature (b) allows for a direct control of the income effect vectors.

The rest of the paper is structured as follows. Section 2 presents the model and notation. Section 3 presents the tools that permit us to analyze the effects of the asset reallocation. In section 4 we obtain two linear independence results derived from the description of the economy. Section 5 deals with the marginal utilities of income of the agents when markets are incomplete. Section 6 presents a technical result on linear algebra, and section 7 completes the proof.

2. The Model

We consider a multigood, two-period ($t = 0, 1$), exchange economy under uncertainty in which one *state* s from a finite *set of states* $\mathcal{S} = \{0, 1, \ldots, S\}$ occurs at date 1. There is a finite set $\mathcal{I} = \{0, 1, \ldots, I\}$ of two-period lived agents who consume only at date 1 and reallocate their income across states by trading securities at date 0. The set of *commodities* is $\mathcal{L} = \{0, 1, \ldots, L\}$. Since there are $L + 1$ commodities available in each state $s \in \mathcal{S}$, the *commodity space* is $\mathbb{R}^n$ with $n = (L + 1)(S + 1)$.

Each agent $i \in \mathcal{I}$ is described by (i) a *consumption set* $X^i \subset \mathbb{R}^n$, (ii) an *initial endowment* vector of the $L + 1$ goods in each state s, $\omega^i := (\omega^i_0, \omega^i_1, \ldots, \omega^i_S)$, where $\omega^i_s := (\omega^i_{0s}, \omega^i_{1s}, \ldots, \omega^i_{Ls})$ and ω^i_{ls} denotes the endowment of commodity $l \in \mathcal{L}$ that agent i has in state s, and (iii) a *utility function* $u^i : X^i \to \mathbb{R}$ defined over consumption bundles $x^i := (x^i_0, x^i_1, \ldots, x^i_S) \in X^i$, where $x^i_s := (x^i_{0s}, x^i_{1s}, \ldots, x^i_{Ls})$ and x^i_{ls} denotes the consumption of commodity l by agent i in state s. Let $z^i := [x^i - \omega^i]$ denote the excess demand of agent i. Let $\omega := (\omega^0, \omega^1, \ldots, \omega^I) \in \mathbb{R}^{n(I+1)}$ and $x := (x^0, x^1, \ldots, x^I) \in \mathbb{R}^{n(I+1)}$ denote, respectively, a vector of endowments and an *allocation of commodities.*

There is a set $\mathcal{A} = \{0, 1, \ldots, A\}$ of *inside real assets* which pay a return in terms of commodity 0 in each state $s \in \mathcal{S}$ denoted, for $a \in \mathcal{A}$, by $r_a(s) \in \mathbb{R}$. For $a \in \mathcal{A}$, we define $r_a := (r_a(0), r_a(1), \ldots, r_a(S)) \in \mathbb{R}^{S+1}$, the payoff vector of asset a. For $s \in \mathcal{S}$, we define $r(s) := (r_0(s), r_1(s), \ldots, r_A(s)) \in \mathbb{R}^{A+1}$, the vector of asset returns in state s. Let

$$R := \begin{bmatrix} [r(0)]^T \\ [r(1)]^T \\ \vdots \\ [r(S)]^T \end{bmatrix} = [r_0 \quad r_1 \quad \ldots \quad r_A] = \begin{bmatrix} r_0(0) & r_1(0) & \ldots & r_A(0) \\ r_0(1) & r_1(1) & \ldots & r_A(1) \\ \vdots & \vdots & & \vdots \\ r_0(S) & r_1(S) & \ldots & r_A(S) \end{bmatrix}$$

be the corresponding *matrix of returns*, of dimension $(S+1) \times (A+1)$.

We denote the quantity of asset a held by agent i by $\theta_a^i \in \mathbb{R}$, a *portfolio* of agent i by $\theta^i := (\theta_0^i, \theta_1^i, \ldots, \theta_A^i) \in \mathbb{R}^{A+1}$, and an allocation of assets by $\theta := (\theta^0, \theta^1, \ldots, \theta^I) \in \mathbb{R}^{(A+1)(I+1)}$.

We assume throughout the paper that

ASSUMPTION A.1. Endowments and Preferences of the Agents: For each $i \in \mathcal{I}$; (i) $\omega^i \in \mathbb{R}^n_{++}$, (ii) u^i is C^2, strictly increasing, and differentiably strictly quasi-concave, and (iii) if $U^i(k) := \{y \in \mathbb{R}^n \ : \ u^i(y) \geq u^i(k)\}$, then $U^i(k) \subset \mathbb{R}^n_{++}$ for each $k \in \mathbb{R}^n_{++}$.

ASSUMPTION A.2. Asset Structure: (i) R has full column rank, (ii) there exists a portfolio $\theta \in \mathbb{R}^{A+1}$ such that $R \cdot \theta > \underline{0}$,[2] (iii) $A < S$, and (iv) each set of $A+1$ rows of R is linearly independent.

Assumptions A.1 and A.2 are standard. Assumption A.1 (iii) says that the closure of the indifference curves of each agent does not intersect the boundary of $\mathbb{R}^n_+$. Also, we have assumed that the asset market is incomplete, Assumption A2 (iii), so that if $\langle R \rangle := \{\tau \in \mathbb{R}^{S+1} \ : \ \tau = R \cdot \theta, \theta \in \mathbb{R}^{A+1}\}$ then $\langle R \rangle \subset \mathbb{R}^{S+1}$ with $\langle R \rangle \neq \mathbb{R}^{S+1}$, i.e., the asset structure does not allow agents to transfer income fully across states.

To ease part of the proof we assume that utilities satisfy a vN-M utility form.

ASSUMPTION A.3. Additively Separable Utilities: For each agent $i \in \mathcal{I}$, there is a Bernoulli utility function $v^i : \mathbb{R}^{L+1}_+ \to \mathbb{R}$, and a

[2] When comparing two vectors x and y of the same dimension we use the symbols "$<$", and "$\leq$" to indicate $x_k \leq y_k$ for each k but $x \neq y$, and $x_k \leq y_k$ for each k respectively.

probability distribution $(\rho^i_s)_{s\in\mathcal{S}} \in \mathbb{R}^{S+1}_+$, such that $u^i(x^i) := \sum_{s\in\mathcal{S}}\rho^i_s v^i(x^i_s)$ for each $x^i \in X^i$.

We denote the vector of *commodity prices* by $p := (p_0, p_1, \ldots, p_S) \in \mathbb{R}^n_+$, where $p_s := (p_{0s}, p_{1s}, \ldots, p_{Ls})$ and p_{ls} is the price of commodity l in state s. Let $q := (q_0, q_1, \ldots, q_A) \in \mathbb{R}^{A+1}$ denote the vector of *asset prices*, where q_a is the price of asset a. We choose commodity 0 as *numeraire* and normalize its price to 1 in each state $s \in \mathcal{S}$. Analogously, we normalize the price of asset 0 by setting $q_0 := 1$. Let $\mathcal{P} := \{p \in \mathbb{R}^n_+ \ : \ p_{0s} = 1 \text{ for each } s \in \mathcal{S}\}$ and $\mathcal{Q} := \{q \in \mathbb{R}^{A+1} \ : \ q_0 = 1\}$ denote, respectively, the normalized price domain for commodities and for assets.

For two vectors $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_w)$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_w)$, with $w \in \mathbb{N}$, where, for each $k = 1, \ldots, w$, α_k and β_k lie in some Euclidean space such that the product $\alpha_k \cdot \beta_k$ is well defined, we define the *box* product $\alpha \,\square\, \beta := (\alpha_1 \cdot \beta_1, \alpha_2 \cdot \beta_2, \ldots, \alpha_w \cdot \beta_w)$.

For a commodity price vector $p \in \mathcal{P}$ and an asset price vector $q \in \mathcal{Q}$, we define the *contingent spot-financial market* budget set of agent i by

$$\mathcal{B}^i(p, q) := \{(x^i, \theta^i) \in X^i \times \mathbb{R}^{A+1} \ : \ q \cdot \theta^i \leq 0, p \,\square\, (x^i - \omega^i) \leq R \cdot \theta^i\}.$$

Since we will obtain a generic result, we have to work with a set of economies rather than with only one. Such a set is obtained via a parameterization of the economy based on both fundamentals, utilities and endowments. So, the *characteristics* of the economy are summarized by the collection of utility functions and endowment vectors of the agents; let $(u, \omega) := (u^0, \ldots, u^I, \omega^0, \ldots, \omega^I)$. We denote the space of utility functions by $\mathcal{U}$ and the space of endowment vectors by Ω. Let $\Gamma := \mathcal{U} \times \Omega$ denote the space of economies that we consider; i.e., we obtain a parameterized family of economies. We say that a set of economies is *generic* if it is an open set of full measure in the space Γ.

Now we can define equilibrium

DEFINITION 1 (CE). *We say that $(x^*, \theta^*, p^*, q^*)$ is a Competitive Equilibrium, (CE) of the economy $(u, \omega) \in \Gamma$ if*

(i) (a) $\sum_i (x^{i^*} - \omega^i) \leq \underline{0}$;
(b) $\sum_i \theta^{i^*} = \underline{0}$,
(ii) *for each $i \in \mathcal{I}$;*

(a) $(x^{i^*}, \theta^{i^*}) \in \mathcal{B}^i(p^*, q^*)$;
(b) *if* $u^i(x^i) > u^i(x^{i^*})$ *for some* x^i *and some* θ^i, *then* $(x^i, \theta^i) \notin \mathcal{B}^i(p^*, q^*)$.

For $i \in \mathcal{I}$, let $(f^i, \zeta^i) : \mathcal{P} \times \mathcal{Q} \to X^i \times \mathbb{R}^{A+1}$ denote the function defined by the fact that, for each $(p, q) \in \mathcal{P} \times \mathcal{Q}$, $(f^i(p, q), \zeta^i(p, q))$ solves the problem

$$\max_{\{(x^i, \theta^i)\}} u^i(x^i) \text{ subject to } q \cdot \theta^i \leq 0 \text{ and } p \,\square\, (x^i - \omega^i) \leq R \cdot \theta^i.$$

Let the function $F : \mathcal{P} \times \mathcal{Q} \to \mathbb{R}^n$ defined by $F(p, q) := \sum_i [f^i (p, q) - \omega^i]$ for each $(p, q) \in \mathcal{P} \times \mathcal{Q}$ denote the *aggregate excess demand* function for goods with spot-financial markets. Also, let the function $\Psi : \mathcal{P} \times \mathcal{Q} \to \mathbb{R}^{A+1}$ defined by $\Psi(p, q) := \sum_i \zeta^i(p, q)$ for each $(p, q) \in \mathcal{P} \times \mathcal{Q}$ denote the *aggregate excess demand* function for assets with estoy en e spot-financial markets.

For a commodity price vector $p \in \mathcal{P}$ and a portfolio $\theta^i \in \mathbb{R}^{A+1}$, we define the *contingent spot market* budget set of agent i by

$$\widetilde{\mathcal{B}}^i(p, \theta^i) := \{x^i \in X^i : p \,\square\, (x^i - \omega^i) \leq R \cdot \theta^i\}.$$

For $i \in \mathcal{I}$, let $g^i : \mathcal{P} \times \mathbb{R}^{A+1} \to X^i$ denote the function defined, for each $(p, \theta^i) \in \mathcal{P} \times \mathbb{R}^{A+1}$, by

$$g^i(p, \theta^i) := \arg\ \max \left\{ u^i(x^i) : x^i \in \widetilde{\mathcal{B}}^i(p, \theta^i) \right\}.$$

DEFINITION 2 (SM-CE). *Given an allocation of assets* $\theta \in \mathbb{R}^{(A+1)(I+1)}$ *such that* $\sum_i \theta^i = \underline{0}$, *we say that* (x^{**}, p^{**}) *is a Spot Market Competitive Equilibrium (SM-CE) of the economy* $(u, \omega) \in \Gamma$ *if*

(i) $\sum_i (x^{i^{**}} - \omega^i) \leq \underline{0}$,
(ii) *for each* $i \in \mathcal{I}$; $x^{i^{**}} = g^i(p^{**}, \theta^i)$.

Let the function $G : \mathcal{P} \times \mathbb{R}^{(A+1)(I+1)} \to \mathbb{R}^n$ defined by $G(p, \theta) := \sum_i [g^i(p, \theta^i) - \omega^i]$ for each $(p, \theta) \in \mathcal{P} \times \mathbb{R}^{(A+1)(I+1)}$ denote the *aggregate excess demand* function for goods with spot markets.

REMARK 1. Consider a pair $(p, q) \in \mathcal{P} \times \mathcal{Q}$. For each $i \in \mathcal{I}$, we have that if $(x^i, \theta^i) \in \mathcal{B}^i(p, q)$, then $x^i \in \widetilde{\mathcal{B}}^i(p, \theta^i)$. Therefore, if

$(x^*, \theta^*, p^*, q^*)$ is a CE, then (x^*, p^*) is a SM-CE for the asset allocation θ^*.

REMARK 2. By invoking Walras' law, we shall consider markets for just L commodities in each state, and for A assets; commodity 0 and asset 0 correspond to the "dropped" markets. Therefore, for $i \in \mathcal{I}$, we denote by $\hat{f}^i = (f^i_{10}, \dots, f^i_{L0}, \dots, f^i_{1S}, \dots, f^i_{LS})$ the truncation of f^i, and by $\hat{F} = (F_{10}, \dots, F_{L0}, \dots, F_{1S}, \dots, F_{LS})$ and $\hat{\Psi} = (\Psi_1, \dots, \Psi_A)$, respectively, the truncation of F and the truncation of Ψ, each of them being defined on the normalized price domain $\mathcal{P} \times \mathcal{Q}$. Analogously, let $\hat{g}^i = (g^i_{10}, \dots, g^i_{L0}, \dots, g^i_{1S}, \dots, g^i_{LS})$ and $\hat{G} = (G_{10}, \dots, G_{L0}, \dots, G_{1S}, \dots, G_{LS})$ denote, respectively, the truncation of g^i and the truncation of G, both of them being defined on the normalized price domain $\mathcal{P}$. Let $\hat{x}^i = (x^i_{10}, \dots, x^i_{L0}, \dots, x^i_{1S}, \dots, x^i_{LS})$, $\hat{\omega}^i = (\omega^i_{10}, \dots, \omega^i_{L0}, \dots, \omega^i_{1S}, \dots, \omega^i_{LS})$, and $\hat{z}^i = (z^i_{10}, \dots, z^i_{L0}, \dots, z^i_{1S}, \dots, z^i_{LS})$ denote, respectively, the truncation of x^i, the truncation of ω^i, and the truncation of z^i.

The notion of optimality used is the benchmark for incomplete asset markets. It applies the concept of Pareto efficiency to the economy above, but imposing that any alternative allocation be traded in the existing markets. This yields the criterion of constrained Pareto optimality, due to Stiglitz (1982), and Newbery and Stiglitz (1982).

DEFINITION 3 (CS). *An allocation (x, θ) is Constrained Suboptimal, CS, if there exists an alternative allocation $(\tilde{x}, \tilde{\theta})$, and a price vector $p \in \mathcal{P}$ such that*

(i) $(\tilde{x}, p)$ *is a SM-CE for the asset allocation* $\tilde{\theta}$,
(ii) (a) $u^i(\tilde{x}^i) \geq u^i(x^i)$ *for each* $i \in \mathcal{I}$;
(b) $u^j(\tilde{x}^j) > u^j(x^j)$ *for some* $j \in \mathcal{I}$.

So, an allocation is *CS* if a (benevolent) "central planner" is able, by redistributing agents' assets and by allowing agents to retrade only goods, to induce a new equilibrium allocation of goods that Pareto dominates the original allocation. Of course, there will be also a new supporting equilibrium price vector associated with the new equilibrium allocation, as stated in Definition 3.

We can now state the GP result.

THEOREM 1 (GP). *Assume A.1, A.2, and A.3, and that* $0 < 2L \leq I < LS$, *and* $A \geq 1$. *Then there exists a generic set of economies* $\tilde{\Gamma} \subset \Gamma$ *such that, for each economy* $(u, \omega) \in \tilde{\Gamma}$, *each CE is CS.*

3. Preliminaries

The objective of this section is to present the problem as one of intervention by a "central planner" and to introduce the tools which will allow us to interpret its effects on the agents' welfare. As a first step, we present two results on the generic regularity of the set of economies described.

To do this, we need first to set a notational convention. For any function H parameterized by the fundamentals of the economy (u, ω), H_y denotes the function H such that parameter $y \in \{u, \omega, (u, \omega)\}$ is fixed; e.g., $(\hat{F}, \hat{\Psi})_{(u,\omega)}$ denotes the (truncated) aggregate excess demand function for goods and assets for the specific economy $(u, \omega) \in \Gamma$, and $(\hat{F}, \hat{\Psi})_u$ denotes the (truncated) aggregate excess demand function for goods and assets for an economy with a fixed utility parameter $u \in \mathcal{U}$ when the endowment $\omega \in \Omega$ is allowed to vary.

PROPOSITION 1. *Generic Regularity: Assume A.1, A.2 (i) and (ii), then, for each $u \in \mathcal{U}$, there exists a generic set $\varrho(u) \subset \Omega$ such that, for each $\omega \in \varrho(u)$, $(\hat{F}, \hat{\Psi})_u$ is a continuously differentiable function with respect to ω.*

PROOF. (GP) ■

Let $\Gamma_1 := \{(u, \omega) \in \Gamma : u \in \mathcal{U}\}, \omega \in \varrho(u)$ denote the generic set of economies identified in Proposition 1.

Since, by Proposition 1, equilibria are *locally isolated* (i.e., for each equilibrium, there is no other equilibrium arbitrarily close to it, so that each equilibrium depends in a continuous manner on the fundamentals of the economy), utility functions can be perturbed by the addition of a quadratic term in a way such that the linear term subsequently added to the vector of the first derivatives amounts to zero at the equilibrium allocation. Therefore, the perturbation leaves unaffected demand but it changes the matrix of second derivatives of the utility function. Using this fact, it can be shown that any perturbation of each of the derivatives $D_p\hat{g}^i$, $i \in \mathcal{I}$, by the addition of a symmetric matrix, can be induced by adding a suitably chosen quadratic term to the utility function of agent i.[3] GP use this result to prove the next proposition.

PROPOSITION 2. *Generic Strong Regularity: Assume A.1, A.2 (i) and (ii), then there exists a generic set of economies $\Gamma_2 \subset \Gamma_1$ such that, for*

[3] See, e.g., Geanakoplos and Polemarchakis (1980).

each $(u, \omega) \in \Gamma_2$ *and each feasible asset allocation* $\theta \in \mathbb{R}^{(A+1)(I+1)}$, *the Jacobian matrix* $D_p \hat{G}(p^*, \theta)$, *evaluated at the SM-CE prices* $p^* \in \mathcal{P}$ *associated with* θ, *is invertible.*

PROOF. (GP) ■

We will now introduce a (benevolent) "central planner", who reallocates the existing assets before trade takes place. After that intervention, agents are allowed to trade in the markets for goods to the point where a new equilibrium in the commodity markets is achieved. However, they are not allowed to retrade the portfolio they were assigned; i.e., the original equilibrium is a CE and the new equilibrium is a SM-CE associated with the new asset allocation. We must show that, for a generic set of economies, the allocation of commodities induced by the new asset reallocation is Pareto improving.

The asset redistribution directly affects the income of the agents and, since more than a single good is traded, it also changes commodity prices in the spot markets at date 1. Both types of effects change the budget sets of the agents and therefore their consumption possibilities. However, intuitively we can see that the direct effect of any feasible asset reallocation on the income of the agents does not permit a Pareto improvement since only a redistribution of a fixed amount of income takes place, so that improving the welfare of an agent necessarily implies reducing that of another. Therefore, we should concentrate on analyzing the effects on welfare due to the *price effect* that results from the reallocation of assets.

Given a pair $(p, q) \in \mathcal{P} \times \mathcal{Q}$, consider the optimization problem of an agent $i \in \mathcal{I}$

$$\max_{\{(x^i, \theta^i)\}} u^i(x^i) \text{ subject to } q \cdot \theta^i \leq 0 \text{ and } p \,\square\, (x^i - \omega^i) \leq R \cdot \theta^i. \quad \text{(P)}$$

The first order conditions for an interior solution (x^i, θ^i) are

$$\mu^i [q]^T = \lambda^i \cdot R, \quad \text{(c1)}$$

$$D_{x^i} u^i(x^i) = p \,\square\, \lambda^i, \quad \text{(c2)}$$

where μ^i and $\lambda^i = (\lambda^i_0, \lambda^i_1, \ldots, \lambda^i_S)$ are, respectively, the Lagrange multipliers corresponding to the budget constraints on assets and on the spot market for agent i in each state s.

From (c2) above, by noting that $\mathrm{d}u^i(x^i) := D_{x^i} u^i(x^i) \cdot \mathrm{d}x^i$, the change in utility of agent i due to a marginal change in his consumption plan is

$$\mathrm{d}u^i(x^i) = \lambda^i \cdot [p \,\square\, \mathrm{d}x^i] \tag{1}$$

Now we can consider the changes induced by such an asset perturbation on the agents' consumption plans. So, by taking infinitesimal perturbations of θ^i that induce changes on x^i and on p, and by computing the total differential of the contingent spot market budget constraint of agent i at the solution, we have

$$p \,\square\, \mathrm{d}x^i = R \cdot \mathrm{d}\theta^i - \mathrm{d}p \,\square\, (x^i - \omega^i), \tag{2}$$

a condition that must be satisfied by the changes induced by the asset reallocation. Then, by combining equations (1) and (2), we obtain

$$\mathrm{d}u^i(x^i) = \lambda^i \cdot R \cdot \mathrm{d}\theta^i - \lambda^i \cdot (x^i - \omega^i) \,\square\, \mathrm{d}p. \tag{3}$$

The first element in equation (3) above reflects the direct effect of the asset reallocation on the utility of agent i due to a perturbation of his income, and the second reflects the contribution due to the change in relative prices. We turn now to a more detailed analysis of this *price effect*.

Consider an initial CE $(x^*, \theta^*, p^*, q^*)$ of an economy $(u, \omega) \in \Gamma_2$. By noting Remark 1 and that the budget constraints of problem (P) above hold with equality at the solution, given assumption A.1, we have that $\hat{G}(p^*, \theta^*) = \underline{0}$. Now, by considering infinitesimal perturbations on p^* and on θ^*, and by computing the total differential, we obtain

$$D_p\hat{G}(p^*, \theta^*) \cdot \mathrm{d}p + D_\theta\hat{G}(p^*, \theta^*) \cdot \mathrm{d}\theta = \underline{0}.$$

From the Strong Regularity result, Proposition 2, we know that, for economies $(u, \omega) \in \Gamma_2$, $D_p\hat{G}(p^*, \theta^*)$ is invertible so that, by applying the Implicit Function Theorem,

$$\mathrm{d}p = -\left[D_p\hat{G}(p^*, \theta^*)\right]^{-1} \cdot D_\theta\hat{G}(p^*, \theta^*) \cdot \mathrm{d}\theta \tag{4}$$

holds in a neighborhood of the initial SM-CE (x^*, p^*) associated with θ^*. Hence, our problem has been reduced to specifying an asset perturbation where the change in utility of each agent $i \in \mathcal{I}$ is given by (3), and the change in prices is determined by the matrix $D_\theta\hat{G}(p^*, \theta^*)$, of dimension $L(S+1) \times (A+1)(I+1)$, that appears in equation (4).

For the original SM-CE (x^*, p^*) associated with the initial asset allocation θ^*, by applying equation (3) combined with equation (4) to each agent $i \in \mathcal{I}$ (considering truncated bundles), we obtain the matrix equation

$$\mathrm{d}u(x^*) = \left(\widetilde{\lambda}^* \cdot \widetilde{R} + \widetilde{\lambda}^* \cdot \psi(x^*) \square \left[D_p \hat{G}(p^*, \theta^*)\right]^{-1} \cdot D_\theta \hat{G}(p^*, \theta^*)\right) \cdot \mathrm{d}\theta, \quad (5)$$

where $\mathrm{d}u(x^*) := (\mathrm{d}u^0(x^{0*}), \mathrm{d}u^1(x^{1*}), \ldots, \mathrm{d}u^I(x^{I*})) \in \mathbb{R}^{I+1}$, and

$$\widetilde{\lambda}^* := \begin{bmatrix} [\lambda^{0*}]^T & [\underline{0}]^T & \ldots & [\underline{0}]^T \\ [\underline{0}]^T & [\lambda^{1*}]^T & \ldots & [\underline{0}]^T \\ \vdots & \vdots & & \vdots \\ [\underline{0}]^T & [\underline{0}]^T & \ldots & [\lambda^{I*}]^T \end{bmatrix}, \ \widetilde{R} := \begin{bmatrix} R & \underline{0} & \ldots & \underline{0} \\ \underline{0} & R & \ldots & \underline{0} \\ \vdots & \vdots & & \vdots \\ \underline{0} & \underline{0} & \ldots & R \end{bmatrix}, \text{ and}$$

$$\psi(x^*) := \begin{bmatrix} \hat{z}^{0*} & \underline{0} & \ldots & \underline{0} \\ \underline{0} & \hat{z}^{1*} & \ldots & \underline{0} \\ \vdots & \vdots & & \vdots \\ \underline{0} & \underline{0} & \ldots & \hat{z}^{I*} \end{bmatrix}$$

with $\widetilde{\lambda}$ being of dimension $(I+1)\times(S+1)(I+1)$, $\widetilde{R}$ being of dimension $(S+1)(I+1) \times (A+1)(I+1)$, and $\psi(x^*)$ being of dimension $L(S+1)(I+1) \times (I+1)$.

For the given SM-CE (x^*, p^*), and for θ^*, let $\mathcal{O}(x^*, p^*, \theta^*)$ denote the matrix, of dimension $(I+1) \times (A+1)(I+1)$, defined by

$$\mathcal{O}(x^*, p^*, \theta^*) := \widetilde{\lambda}^* \cdot \psi(x^*) \square \left[D_p \hat{G}(p^*, \theta^*)\right]^{-1} \cdot D_\theta \hat{G}(p^*, \theta^*). \quad (6)$$

Also, for $i \in \mathcal{I}$, let $V_s^i(p^*) = (V_{1s}^i(p^*), \ldots, V_{Ls}^i(p^*)) \in \mathbb{R}^L$ denote the vector of *income effects* of agent i in state s at p^*; i.e.,

$$V_{ls}^i(p^*) := \frac{\partial \hat{g}_{ls}^i}{\partial w_s^i}(p^*, \theta^{i*})$$

where $w_s^i := r(s) \cdot \theta^i$ for $i \in \mathcal{I}$ and $s \in \mathcal{S}$, the change, at the given SM-CE, in the demand for good $l \in \mathcal{L}\backslash\{0\}$ by agent i in state s due an infinitesimal change of his income in that state. We set $V^i(p^*) := (V_0^i(p^*), \ldots, V_S^i(p^*)) \in \mathbb{R}^{L(S+1)}$. Now, since, for $i \in \mathcal{I}$, $l \in \mathcal{L}\backslash\{0\}$, $s \in \mathcal{S}$, and $a \in \mathcal{A}$, we have

$$\frac{\partial \hat{g}^i_{ls}}{\partial \theta^i_a}(p^*, \theta^{i^*}) = r_a(s)V^i_{ls}(p^*),$$

the matrix $D_\theta \hat{g}^i(p^*, \theta^{i^*})$, of dimension $L(S+1) \times (A+1)(I+1)$, can be written as

$$D_\theta \hat{g}^i(p^*, \theta^{i^*}) = [\underline{0} \ldots \underline{0}\; r_0 \Box V^i(p^*)\; r_1 \Box V^i(p^*) \ldots\; r_A \Box V^i(p^*) \underline{0} \ldots\; \underline{0}] \quad (7)$$

where the non-null columns correspond to the changes in the demand of agent i due to the changes in the portfolio of that agent while the null vectors correspond to the changes induced by the variations in the portfolio of agents other than i.

We turn now to specify the asset reallocation that we consider.

The *proposed asset reallocation* is such that agent 0 gifts asset 0 to each agent $j \in \mathcal{I}\backslash\{0\}$ and gifts asset 1 to agent 1. Let $\tau^j_a \in \mathbb{R}$ denote a transfer of asset a that agent $j \in \mathcal{I}\backslash\{0\}$ receives from agent 0. The changes in asset holdings associated with the asset reallocation are then denoted by $\Delta\theta = (\Delta\theta^0, \Delta\theta^1, \ldots, \Delta\theta^I) \in \mathbb{R}^{(A+1)(I+1)}$ and specified by

$$\Delta\theta^0 := (-\sum_{j=1}^{I} \tau^j_0, -\tau^1_1, 0, \ldots, 0), \quad \Delta\theta^1 := (\tau^1_0, \tau^1_1, 0, \ldots, 0),$$

and by

$$\Delta\theta^m := (\tau^m_0, 0, 0, \ldots, 0) \text{ for each } m \in \mathcal{I}\backslash\{0, 1\},$$

so that the vector $\Delta\theta$ has $I+1$ non-zero entries that can be set "independently". Let $\tau := (\tau^1_0, \tau^2_0, \ldots, \tau^I_0, \tau^1_1)$ denote a vector of *asset transfers* that must be chosen to lie in the *space of transfers* $\mathcal{T} := \mathbb{R}^{I+1}$.

REMARK 3. By using the proposed asset reallocation, for each $\Delta\theta \in \mathbb{R}^{(A+1)(I+1)}$, there is a unique $\tau \in \mathcal{T}$ that fully specifies $\Delta\theta$.

With this intervention, by noting (7), we obtain the changes induced in the demand of the agents:

$$\text{(a)} \quad D_\theta \hat{g}^0(p^*, \theta^{0^*}) \cdot \Delta\theta = -r_0 \Box V^0(p^*) \sum_{j=1}^{I} \tau^j_0 - r_1 \Box V^0(p^*) \tau^1_1,$$

(b) $D_\theta \hat{g}^1(p^*, \theta^{1*}) \cdot \Delta\theta = r_0 \,\square\, V^1(p^*)\tau_0^1 + r_1 \,\square\, V^1(p^*)\tau_1^1,$

(c) $D_\theta \hat{g}^m(p^*, \theta^{m*}) \cdot \Delta\theta = r_0 \,\square\, V^m(p^*)\tau_0^m$ for each $m \in \mathcal{I} \setminus \{0, 1\}$.

Then, since $D_\theta \hat{G}(p^*, \theta^*) \cdot \Delta\theta = \sum_i D_\theta \hat{g}^i(p^*, \theta^{i*}) \cdot \Delta\theta$, we obtain, for an asset reallocation $\Delta\theta$ specified by means $\tau \in \mathcal{T}$,

$$D_\theta \hat{G}(p^*, \theta^*) \cdot \Delta\theta = \mathrm{A}(p^*) \cdot \tau, \tag{8}$$

where $\mathrm{A}(p^*)$ denotes the matrix, of dimension $L(S+1) \times (I+1)$, specified by

$$\mathrm{A}(p^*) := \Big[r_0 \,\square\, [V^1(p^*) - V^0(p^*)] \;\; r_0 \,\square\, [V^I(p^*) - V^0(p^*)] \\ r_1 \,\square\, [V^1(p^*) - V^0(p^*)] \Big]. \tag{9}$$

From equation (5), using the matrix specified in (6), and taking into account the proposed reallocation, we have that

$$\mathrm{d}u(x^*) = \left(\widetilde{\lambda}^* \cdot \widetilde{R} + \mathcal{O}(x^*, p^*, \theta^*) \right) \cdot \Delta\theta.$$

So, our objective is to analyze whether for a generic set of economies the rank of matrix $(\widetilde{\lambda}^* \cdot \widetilde{R} + \mathcal{O}(x^*, p^*, \theta^*))$, of dimension $(I + 1) \times (A+1)(I+1)$, equals $(I+1)$ so that, by choosing appropriately the vector $\Delta\theta$, any $\mathrm{d}u(x^*) \in \mathbb{R}^{I+1}$ can be generated. A standard argument shows that the rank of matrix $\widetilde{\lambda}^* \cdot \widetilde{R}$ cannot be $I+1$ since it only captures the effect of a pure redistribution of income. It follows that to prove Theorem T, it suffices to show that matrix $\mathcal{O}(x^*, p^*, \theta^*)$ has rank $I+1$ for a generic set of economies. By noting Remark 3 and by using (6) together with (8), we obtain that, for each $\Delta\theta \in \mathbb{R}^{(A+1)(I+1)}$, there is a unique $\tau \in \mathcal{T}$ such that

$$\mathcal{O}(x^*, p^*, \theta^*) \cdot \Delta\theta = \widetilde{\lambda}^* \cdot \psi(x^*) \,\square\, \left[D_p \hat{G}(p^*, \theta^*) \right]^{-1} \cdot \; \mathrm{A}(p^*) \cdot \tau.$$

Then, it suffices to show that the matrix $\Phi(x^*, p^*, \theta^*)$, of dimension $(I+1) \times (I+1)$, specified by

$$\Phi(x^*, p^*, \theta^*) := \widetilde{\lambda}^* \cdot \psi(x^*) \,\Box\, \left[D_p \hat{G}(p^*, \theta^*) \right]^{-1} \cdot \; \mathrm{A}(p^*),$$

where $\mathrm{A}(p^*)$ is the matrix specified in (9), has rank $I+1$ for a generic set of economies. To prove this, we will show that, generically, there is no $\delta \in \Delta^{I+1} := \{y \in \mathbb{R}_+^{I+1} \ : \ \sum_k y_k = 1\}$ such that $\delta \cdot \Phi(x^*, p^*, \theta^*) = [\underline{0}]^T$.

The proof will be completed in two steps.

STEP 1. We will show, in Proposition 4, that generically any matrix obtained by dropping from $\mathrm{A}(p^*)$ the vectors that correspond to any state has rank $I+1$.

STEP 2. We will show in section 7 that, for $\delta \in \Delta^{I+1}$, by suitably perturbing (u, ω), we can alter as we wish at least LS entries (that correspond to at least S states) of the vector $\delta \cdot \widetilde{\lambda}^* \cdot \psi(x^*) \,\Box\, [D_p \hat{G}(p^*, \theta^*)]^{-1}$, leaving $[D_p \hat{G}(p^*, \theta^*)]^{-1}$ unchanged. To do so, we use a result from linear algebra provided in Lemma L, together with (i) the result on linear independence given in Proposition 3, and (ii) the property in Proposition 5, whereby there is a set of $L+1$ agents $\{i_0, i_1, \ldots, i_L\} \subset \mathcal{I}$, such that, given $\delta := (\delta_{i_0}, \delta_{i_1}, \ldots, \delta_{i_L}) \in \Delta^{L+1}$, generically, $0 \neq \delta_{i_0} \cdot \lambda_s^{i_0 *} \neq \delta_{i_m} \cdot \lambda_s^{i_m *}$ for at least S states, for each $m \in \{1, 2, \ldots, L\}$.

4. Linear Independence of the Income Effects

In this section we obtain two properties of linear independence that the set of vectors $\{V^0, V^1, \ldots, V^I\}$ generically satisfies. These results require that $L > 0$ and that preferences not be quasi-linear since otherwise income effects are absent.

PROPOSITION 3. *Assume A.1, A.2 (i) and (ii), then, for each subset of* $L+1$ *agents,* $\{i_0, i_1, \ldots, i_L\} \subset \mathcal{I}$, *and for each* $s \in \mathcal{S}$, *the set of vectors*

$$\{V_s^{i_1}(p^*) - V_s^{i_0}(p^*), V_s^{i_2}(p^*) - V_s^{i_0}(p^*), \ldots, V_s^{i_L}(p^*) - V_s^{i_0}(p^*)\}$$

is linearly independent, for a CE *price* p^* *of an economy in some generic set* $\Gamma_3 \subset \Gamma$.

PROOF. Consider an arbitrary subset of $L+1$ agents $\{i_0, i_1, \ldots, i_L\} \subset \mathcal{I}$, and a given state $s \in \mathcal{S}$. Define the matrix, of dimension $L \times L$,

$$\Pi_s(p^*) := \left[V_s^{i_1}(p^*) - V_s^{i_0}(p^*) \; V_s^{i_2}(p^*) - V_s^{i_0}(p^*) \; \ldots \; V_s^{i_L}(p^*) - V_s^{i_0}(p^*) \right]$$

and let $\sigma_s : \mathcal{P} \times \mathcal{Q} \times \Delta^L \to \mathbb{R}^{L(S+1)} \times \mathbb{R}^A \times \mathbb{R}^L$ be the function specified by

$$\sigma_s(p, q, \delta) := \left[(\hat{F}, \hat{\Psi})(p, q), \delta \cdot \Pi_s(p^*) \right]$$

for each $(p, q, \delta) \in \mathcal{P} \times \mathcal{Q} \times \Delta^L$. Since utility functions can be perturbed without changing their first derivatives at the equilibrium allocation, we are able to change $V_s^i(p^*)$ for any $i \in \mathcal{I}$ and for any $s \in \mathcal{S}$, maintaining $(\hat{F}, \hat{\Psi})(p^*, q^*)$ unaltered at the CE prices (p^*, q^*). Therefore, by applying a transversality argument, we know that $\sigma_{s(u,\omega)}$ is transverse to zero for each $(u, \omega) \in \Gamma_3$, where $\Gamma_3 \subset \Gamma$ is a generic set. Now, given that the dimension of the range of $\sigma_{s(u,\omega)}$ exceeds that of the domain, by applying the Regular Value Theorem, $\sigma_s^{-1}{}_{(u,\omega)}(\underline{0}) = \emptyset$ for each $(u, \omega) \in \Gamma_3$. Therefore, $\Pi_s(p^*)$ has rank L for a generic set of economies Γ_3.

The result follows by noting that s was chosen arbitrarily. ∎

Notice that, if this property holds, then, for any given $s \in \mathcal{S}$, the set of vectors $\{V_s^{i_1}(p^*) - V_s^{i_0}(p^*), V_s^{i_2}(p^*) - V_s^{i_0}(p^*), \ldots, V_s^{i_L}(p^*) - V_s^{i_0}(p^*)\}$ span $\mathbb{R}^L$.

For $i \in \mathcal{I} \setminus \{0\}$, $a \in \{0, 1\}$, and $s \in \mathcal{S}$, let $\kappa_s^{a,i}(p^*)$ denote the vector, with LS coordinates, obtained from $r_a \,\square\, \left[V^i(p^*) - V^0(p^*)\right]$ by dropping the L coordinates that correspond to state s.

PROPOSITION 4. *Assume A.1, A.2 (i), (ii), and (iv), then, for each $s \in \mathcal{S}$, the set of vectors $\{\kappa_s^{0,1}(p^*), \ldots, \kappa_s^{0,I}(p^*), \kappa_s^{1,1}(p^*)\}$ is linearly independent for a* CE *price p^* of an economy in some generic set $\Gamma_4 \subset \Gamma$.*

PROOF. Pick a state $s \in \mathcal{S}$. We decompose the proof into two steps.

STEP 1. From Assumption A.2 (iv) we know that the rank of each matrix of size $(A+1) \times (A+1)$ obtained by removing from matrix R any set of $S - A$ rows equals $A + 1$. Thus, any set of vectors obtained by considering, for each of the assets in $\mathcal{A}$, the same $A+1$ coordinates of their corresponding vectors of payoffs is linearly independent. Since $A + 1 \geq 2$, we can choose two vectors from the set $\{r_0, r_1, \ldots, r_A\}$

such that they are linearly independent when restricted to any subset, of size $A+1$, of their coordinates. Furthermore, since $S \geq A+1$, we know that these two vectors are also linearly independent when restricted to S arbitrarily chosen coordinates. This result guarantees, in addition, that not all the coordinates of any of the vectors derived in that way equal zero.[4]

Consider, without loss of generality, that $r_0, r_1 \in \mathbb{R}^{S+1}$ are the vectors chosen as described above. It follows that the vectors $\kappa_s^{0,1}(p^*), \kappa_s^{1,1}(p^*)$ are linearly independent since by multiplying r_0 and r_1 by $\left[V^1(p^*) - V^0(p^*)\right]$ according to the box product, the vectors r_0 and r_1 are affected by the same proportion in the same coordinates so that no relative change across the coordinates is induced.

STEP 2. Define the matrix $\Sigma_s(p^*) := \left[\kappa_s^{0,1}(p^*) \ \ldots \ \kappa_s^{0,I}(p^*) \ \kappa_s^{1,1}(p^*)\right]$, of dimension $LS \times (I+1)$. Also, let $\beta_s : \mathcal{P} \times \mathcal{Q} \times \Delta^{I+1} \rightarrow \mathbb{R}^{L(S+1)} \times \mathbb{R}^A \times \mathbb{R}^{LS}$ be the function specified by

$$\beta_s(p,q,\delta) := \left[(\hat{F}, \hat{\Psi})(p,q), \Sigma_s(p^*) \cdot \delta\right]$$

for each $(p,q,\delta) \in \mathcal{P} \times \mathcal{Q} \times \Delta^{I+1}$. Since we can perturb utility functions in a way such that $\left[V^i(p^*) - V^0(p^*)\right]$, and thus also $\kappa_s^{0,i}(p^*)$ and $\kappa_s^{1,1}(p^*)$, for each $i \in \mathcal{I} \setminus \{0\}$, are changed, maintaining $(\hat{F}, \hat{\Psi})(p^*, q^*)$ unaffected at the CE prices (p^*, q^*), we obtain that $\beta_{s(u,\omega)} \pitchfork \underline{0}$ for each $(u,\omega) \in \Gamma_4$, where $\Gamma_4 \subset \Gamma$ is a generic set. Now, since the dimension of the range of $\beta_{s(u,\omega)}$ exceeds that of the domain, for each

[4] In their proof GP claim that by assuming that there exists a portfolio $\theta \in \mathbb{R}^{A+1}$ such that $r(s) \cdot \theta \neq 0$ for each $s \in \mathcal{S}$, and (possibly) by relabelling assets, one obtains that $r_0(s) \neq 0$ for each $s \in \mathcal{S}$. However, easy examples show that such an implication fails to hold. Notice, e.g., that each set of 2 rows of the matrix

$$R = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

is linearly independent, that there exists a portfolio $\theta = (1,1)$ such that $r(s) \cdot \theta \neq 0$ for each $s = 0,1,2$, and that yet not all the coordinates of the two payoff vectors are different from zero. Nevertheless, the proof does not make use either of that assumption or of the result stated by GP.

$(u, \omega) \in \Gamma_4$ there is no $\delta \in \Delta^I$ such that $\Sigma_s(p^*) \cdot \delta = \underline{0}$ so that rank $[\Sigma_s(p^*)] = I + 1$.

The result yields since state s was chosen arbitrarily. ∎

REMARK 4. Since the linear independence property in Proposition 4 is stated for at least LS of the coordinates of the vectors in a set of size $I + 1$, then $I + 1 \leq LS$ appears as a necessary condition for this result to hold. By assuming that $I < LS$, such a condition is satisfied. CKV do not impose an upper bound on the number of agents. They can achieve the constrained suboptimality result so long as they consider a policy with *lump-sum* transfers among agents in period 0. This allows them to control directly the income effect vectors of the agents. Without direct transfers of goods, since the welfare of agents is affected by inducing changes in $L(S + 1)$ relative prices, it is clear that there must be an upper bound on the number of agents. Indeed Mas-Colell (1987) provides an example that shows that Theorem T does not hold if the upper bound on I is removed.

5. Marginal Utility of Income

In this section we obtain two properties of the agents' marginal utilities of income. The first property shows that, generically, the agents' ratios of marginal utilities across states do not coincide, a fact that is strictly derived from the market incompleteness. This fact also drives the result stated in the second property.

PROPOSITION 5. *Assume A.1, A.2 (i), (iii), and (iv), then, at each CE of an economy in a generic set of economies $\Gamma_5 \subset \Gamma$, we have*

$$\frac{\lambda^{i^*}_s}{\lambda^{i^*}_{s'}} \neq \frac{\lambda^{j^*}_s}{\lambda^{j^*}_{s'}}$$

for each $i, j \in \mathcal{I}$, such that $i \neq j$ and each $s, s' \in \mathcal{S}$ such that $s \neq s'$.

PROOF. Define the set $Y_R := \{y \in \mathbb{R}^{S+1} \ : \ y \cdot R = [\underline{0}]^T\}$. From Assumption A.2 (i) and (iii), we know that rank $(R) = A + 1$ and $S + 1 > A + 1$ so that Y_R is generated by a vector space of dimension greater than or equal to one. Fix an arbitrary $\tilde{s} \in \mathcal{S}$, consider a subset of $A + 1$ states $\widehat{\mathcal{S}} \subset \mathcal{S} \setminus \{\tilde{s}\}$, ordered as $s_0, s_1, \ldots, s_A$, set $\widehat{m}_s := 0$ for each $s \notin \widehat{\mathcal{S}}$ such that $s \neq \tilde{s}$, and let $\widehat{m}_{\tilde{s}} \neq 0$ be an arbitrary number. Then, the equation $-\widehat{y}_{\tilde{s}} \cdot r(\tilde{s}) = \sum_{s \in \widehat{\mathcal{S}}} \widehat{y}_s \cdot r(s)$ has a solution since, by

Assumption A.2 (iv), each set of $A+1$ vectors that can be extracted from the set $\{r(0), r(1), \ldots, r(S)\}$ is linearly independent so that they span $\mathbb{R}^{A+1}$. It follows that we can pick a vector $\widehat{y} \in Y_R \setminus \{\underline{0}\}$ even though at least one coordinate is arbitrarily pre-specified.

Now, consider a CE of an economy $(u, \omega) \in \Gamma$. For an agent $i \in \mathcal{I}$, we have that $\mu^{i^*}[q^*]^T = \lambda^{i^*} \cdot R$ specifies the condition (c1) obtained earlier for his optimal choice of an asset portfolio. Take two agents, $i, j \in \mathcal{I}$, $i \neq j$, and two states $s, s' \in \mathcal{S}$, $s \neq s'$. Perturb the utility function of agent i in a way such that a vector denoted by $\eta = (\eta_0, \eta_1, \ldots, \eta_S) \in \mathbb{R}^n$, where $\eta_s := (\eta_{0s}, \eta_{1s}, \ldots, \eta_{Ls})$ for each $s \in \mathcal{S}$, is added to the derivative $D_{x^i} u^i(x^{i^*})$, and, accordingly, the vector λ^{i^*} is perturbed by the addition of a vector $\Delta\lambda^i$. Using condition (c2), obtained earlier, for the optimal choice of goods of agent i we know that the vectors η and $\Delta\lambda^i$ must satisfy the equality $\eta = p^* \,\square\, \Delta\lambda^i$.

By the properties of the set Y_R, it is possible to choose a $\Delta\lambda^i \in M_R$ such that either $\Delta\lambda^i_s \neq 0$ or $\Delta\lambda^i_{s'} \neq 0$. We use this to construct the utility perturbation described above. That perturbation does not affect the optimal choice of assets of agent i since

$$(\lambda^{i^*} + \Delta\lambda^i) \cdot R = \lambda^{i^*} \cdot R + \Delta\lambda^i \cdot R = \lambda^{i^*} \cdot R + [\underline{0}]^T = \lambda^{i^*} \cdot R.$$

In addition, we must compensate the change induced in the demand of agent i. We do this by adding the appropriate amount to his vector of endowments ω^i so as to leave his excess demand unaffected.

Now, define the matrix, of dimension 2×2,

$$\Upsilon^{ij}_{ss'}(p^*) := \begin{bmatrix} \lambda^{i^*}_s & \lambda^{j^*}_s \\ \lambda^{i^*}_{s'} & \lambda^{j^*}_{s'} \end{bmatrix},$$

and let $\varphi^{ij}_{ss'} : \mathcal{P} \times \mathcal{Q} \times \Delta^2 \to \mathbb{R}^{L(S+1)} \times \mathbb{R}^A \times \mathbb{R}^2$ be the function specified by

$$\varphi^{ij}_{ss'}(p, q, \delta) := [(\hat{F}, \hat{\Psi})(p, q), \delta \cdot \Upsilon^{ij}_{ss'}(p^*)]$$

for each $(p, q, \delta) \in \mathcal{P} \times \mathcal{Q} \times \Delta^2$. Since the perturbation of utilities and endowments specified above changes the vector $(\lambda^{i^*}_s, \lambda^{i^*}_{s'})$ leaving $(\hat{F}, \hat{\Psi})(p^*, q^*)$ unaffected at the CE prices (p^*, q^*), then $\varphi^{ij}_{ss'(u,\omega)} \pitchfork \underline{0}$ for each $(u, \omega) \in \Gamma_5$, where $\Gamma_5 \subset \Gamma$ is a generic set. Now, since the dimension of the range of $\varphi^{ij}_{ss'(u,\omega)}$ exceeds that of the domain, by applying the Regular Value Theorem, we obtain that, for such a set of economies, there is no $\delta \in \Delta^2$ such that $\delta \cdot \Upsilon^{ij}_{ss'}(p^*) = [\underline{0}]^T$, i.e., the rank of matrix $\Upsilon^{ij}_{ss'}(p^*)$ is 2, as required. ∎

PROPOSITION 6. *Assume A.1, A.2 (i), (iii), and (iv), then, given* $\delta := (\delta_{i_0}, \delta_{i_1}, \ldots, \delta_{i_L}) \in \Delta^{L+1}$ *such that* $\delta_{i_0} \neq 0$, *there exists a set of* $L+1$ *agents,* $\{i_0, i_1, \ldots, i_L\} \subset \mathcal{I}$, *such that, at each CE of an economy in a generic set* $\Gamma_5 \in \Gamma$, *we have* $0 \neq \delta_{i_0} \lambda_s^{i_0\,*} \neq \delta_{i_m} \lambda_s^{i_m\,*}$ *for at least* S *states, for each* $m \in \{1, 2, \ldots, L\}$.

PROOF. Since, from Assumption A.1, the problem (P) has only interior solutions, then $\lambda_s^{i\,*} \neq 0$ for each $i \in \mathcal{I}$ and each $s \in \mathcal{S}$ at a CE.

Consider an agent $i_0 \in \mathcal{I}$, a subset of states $\widetilde{\mathcal{S}} \subset \mathcal{S}$ such that $\#\widetilde{\mathcal{S}} := S$, and pick a $\delta := (\delta_{i_0}, \delta_{i_1}, \ldots, \delta_{i_L}) \in \Delta^{L+1}$ such that $\delta_{i_0} \neq 0$. By assuming that $I \geq 2L$, we are able to either

(a) Extract from $\mathcal{I} \setminus \{i_0\}$ a set of agents $\{i_1, i_2, \ldots, i_L\} \subset \mathcal{I} \setminus \{i_0\}$ for which $\delta_{i_0} \lambda_s^{i_0\,*} \neq \delta_{i_m} \lambda_s^{i_m\,*}$ for each $m \in \{1, 2, \ldots, L\}$ and each $s \in \widetilde{\mathcal{S}}$, so that the result stated in Proposition 6 holds, or

(b) Extract from $\mathcal{I} \setminus \{i_0\}$ a set of agents $\{j_1, j_2, \ldots, j_L\} \subset \mathcal{I} \setminus \{i_0\}$ such that $\delta_{j_m} \lambda_{\bar{s}}^{j_m\,*} = \delta_{i_0} \lambda_{\bar{s}}^{i_0\,*}$, for each $m \in \{1, 2, \ldots, L\}$, for some $\bar{s} \in \widetilde{\mathcal{S}}$. Then, by using the result stated in Proposition 5, we know that $\frac{\lambda_{\bar{s}}^{j_m\,*}}{\lambda_s^{j_m\,*}} \neq \frac{\lambda_{\bar{s}}^{i_0\,*}}{\lambda_s^{i_0\,*}}$ for each $m \in \{1, 2, \ldots, L\}$, for each $s \in \mathcal{S} \setminus \{\bar{s}\}$, and for each $(u, \omega) \in \Gamma_5$. Therefore, by specifying the set $\bar{\mathcal{S}} := \mathcal{S} \setminus \{\bar{s}\}$, we obtain that $\delta_{i_0} \lambda_s^{i_0\,*} \neq \delta_{i_m} \lambda_s^{i_m\,*}$ for each $m \in \{1, 2, \ldots, L\}$, for each $s \in \bar{\mathcal{S}}$, for each $(u, \omega) \in \Gamma_5$, as required. ∎

6. A Result from Linear Algebra

We will exploit the following Lemma in the next section.

LEMMA 1. *Given a set of* L *non-zero numbers* $\{a_0, a_1, \ldots, a_L\}$ *such that* $a_0 \neq a_m$ *for each* $m \in \{1, 2, \ldots, L\}$, *and a set of* L *linearly independent vectors of dimension* L, $\{v_1, \ldots, v_L\}$, *any vector* $a_0 \sum_{m=1}^{L} \alpha_m v_m - \sum_{m=1}^{L} a_m \alpha_m v_m$, *of dimension* L, *can be generated by suitably choosing the set of numbers* $\{\alpha_1, \alpha_2, \ldots, \alpha_L\}$.

PROOF. (GP) ∎

7. Proof of the Result

In this section we provide the proof of Theorem T by making use of the various arguments presented up to now.

First, we specify the generic set of economies that are strongly regular, Proposition 2, and for which the results stated in Proposition 3, Proposition 4, and Proposition 6 are satisfied as $\hat{\Gamma} := \cap_{k=2}^{5} \Gamma_k$.

Consider a CE $(x^*, \theta^*, p^*, q^*)$ of a given economy $(u, \omega) \in \hat{\Gamma}$. Let us recall that the key procedure to prove Theorem T is to show that the matrix $\Phi(x^*, p^*, \theta^*)$ defined in section 3 has full rank for a generic set of economies. Since we are interested in proving a generic feature, we need to perturb the economy (u, ω). We do this by setting an additive perturbation that induces (u, ω) to move to a neighboring economy, that is,

$$(u, \omega) \longmapsto (u, \omega) + (\Delta u, \Delta \omega),$$

where $\Delta\omega$ and Δu denote, respectively, the perturbation to endowments and the perturbation to utilities.

Let us describe first the perturbation to endowments.

Consider a set of $L+1$ agents $\{i_0, i_1, \ldots, i_L\} \subset \mathcal{I}$ and a subset of states $\widetilde{\mathcal{S}} \subset \mathcal{S}$, $\#\widetilde{\mathcal{S}} = S$, ordered as $s_1, \ldots, s_S$. Set $\{\bar{s}\} := \mathcal{S} \setminus \widetilde{\mathcal{S}}$. Consider, for each $s \in \widetilde{\mathcal{S}}$, an arbitrary set of numbers $\{\gamma_{1s}, \gamma_{2s}, \ldots, \gamma_{Ls}\}$. Then, the vector $\Delta\omega$ is specified as:

(a) $\Delta\omega^i := \underline{0}$ for each $i \notin \{i_0, i_1, \ldots, i_L\}$,

(b) For each $m \in \{1, 2, \ldots, L\}$ and each $s \in \widetilde{\mathcal{S}}$;

$$\begin{aligned} \Delta\omega_s^{i_m} &= \left(\Delta\omega_{0s}^{i_m}, (\Delta\omega_{1s}^{i_m}, \ldots, \Delta\omega_{Ls}^{i_m})\right) : \\ &= \left(\Delta\omega_{0s}^{i_m}, \gamma_{ms}\left[V_s^{i_m}(p^*) - V_s^{i_0}(p^*)\right]^T\right), \end{aligned}$$

and $\Delta\omega_{\bar{s}}^{i_m} := \underline{0}$,

(c) For each $s \in \widetilde{\mathcal{S}}$;

$$\begin{aligned} \Delta\omega_s^{i_0} &= \left(\Delta\omega_{0s}^{i_0}, (\Delta\omega_{1s}^{i_0}, \ldots, \Delta\omega_{Ls}^{i_0})\right) : \\ &= \left(\Delta\omega_{0s}^{i_0}, -\sum_{m=1}^{L} \gamma_{ms}\left[V_s^{i_m}(p^*) - V_s^{i_0}(p^*)\right]^T\right), \end{aligned}$$

and $\Delta\omega_{\bar{s}}^{i_0} := \underline{0}$.

In addition, for each $m \in \{0, 1, \dots, L\}$ and each $s \in \widetilde{\mathcal{S}}$, $\Delta\omega_{0s}^{i_m}$ is specified as to satisfy

$$\Delta\omega_{0s}^{i_m} + \sum_{l=1}^{L} p_{ls}^{*} \Delta\omega_{ls}^{i_m} = 0,$$

so that the income of agent i_m in state $s \in \widetilde{\mathcal{S}}$ remains unaffected.

For $i \in \mathcal{I}$, let $\Delta\hat{z}^i$ denote the change induced in the excess demand of agent i by the perturbation of endowments. We note that the perturbation to endowments does not change the optimal choices of any agent since it leaves unaffected the budget constraints of the agents in each state. Also, it satisfies

(i) $\Delta\hat{z}^i = \underline{0}$ for each $i \notin \{i_0, i_1, \dots, i_L\}$,

(ii) $\Delta\hat{z}_s^{i_m} = \gamma_{ms}\left[V_s^{i_m}(p^*) - V_s^{i_0}(p^*)\right]$ for each $m \in \{1, 2, \dots, L\}$ and each $s \in \widetilde{\mathcal{S}}$,

(iii) $\Delta\hat{z}_s^{i_0} = -\sum_{m=1}^{L} \gamma_{ms}\left[V_s^{i_m}(p^*) - V_s^{i_0}(p^*)\right]$ for each $s \in \widetilde{\mathcal{S}}$, and

(iv) $\Delta\hat{z}_{\bar{s}}^{i_m} = \underline{0}$ for each $m \in \{0, 1, \dots, L\}$.

These changes in the excess demands of the agents translate into a change of the matrix $\psi(x^*)$ which we denote by $\Delta\psi(x^*)$. Then, for an arbitrary vector $\delta := (\delta_0, \delta_1, \dots, \delta_I) \in \Delta^{I+1}$ we obtain the change induced in $\delta \cdot \widetilde{\lambda}^* \cdot \psi(x^*)$ by the specified perturbation on endowments as

$$\delta \cdot \widetilde{\lambda}^* \cdot \Delta\psi(x^*) = \sum_{i=0}^{I} \delta_i {\lambda^{i}}^{*} \cdot \Delta\hat{z}^i = \sum_{m=0}^{L} \delta_{i_m} {\lambda^{i_m}}^{*} \cdot \Delta\hat{z}^{i_m}$$

since $\Delta\hat{z}^i = \underline{0}$ for each $i \notin \{i_0, i_1, \dots, i_L\}$.

Upon substituting for each $\Delta\hat{z}^{i_m}$, we obtain

$$\delta \cdot \widetilde{\lambda}^* \cdot \Delta\psi(x^*) = -\delta_{i_0}\Bigg({\lambda^{i_0}}^{*}_{s_1} \sum_{m=1}^{L} \gamma_{ms_1}\left[V_{s_1}^{i_m}(p^*) - V_{s_1}^{i_0}(p^*)\right]^T \dots [\underline{0}]^T \dots$$

$$\dots \; {\lambda^{i_0}}^{*}_{s_S} \sum_{m=1}^{L} \gamma_{ms_S}\left[V_{s_S}^{i_m}(p^*) - V_{s_S}^{i_0}(p^*)\right]^T \Bigg)$$

$$+\sum_{m=1}^{L}\delta_{i_m}\Bigg(\lambda^{i_m}{}^*_{s_1}\gamma_{ms_1}\left[V^{i_m}_{s_1}(p^*)-V^{i_0}_{s_1}(p^*)\right]^T\ \ldots\ [\underline{0}]^T\ \ldots$$

$$\ldots\ \lambda^{i_m}{}^*_{s_s}\gamma_{ms_S}\left[V^{i_m}_{s_S}(p^*)-V^{i_0}_{s_S}(p^*)\right]^T\Bigg)$$

$$=\Bigg(-\delta_{i_0}\lambda^{i_0}{}^*_{s_1}\sum_{m=1}^{L}\gamma_{ms_1}\left[V^{i_m}_{s_1}(p^*)-V^{i_0}_{s_1}(p^*)\right]^T$$

$$+\sum_{m=1}^{L}\delta_{i_m}\lambda^{i_m}{}^*_{s_1}\gamma_{ms_1}\left[V^{i_m}_{s_1}(p^*)-V^{i_0}_{s_1}(p^*)\right]^T$$

$$\ldots\quad[\underline{0}]^T\quad\ldots$$

$$-\delta_{i_0}\lambda^{i_0}{}^*_{s_S}\sum_{m=1}^{L}\gamma_{ms_S}\left[V^{i_m}_{s_S}(p^*)-V^{i_0}_{s_S}(p^*)\right]^T$$

$$+\sum_{m=1}^{L}\delta_{i_m}\lambda^{i_m}{}^*_{s_S}\gamma_{ms_S}\left[V^{i_m}_{s_S}(p^*)-V^{i_0}_{s_S}(p^*)\right]^T\Bigg),$$

so that there are $S+1$ blocks of L dimensional row vectors of which one block, the one that corresponds to state $\bar{s}$, is a vector of zeros.

We recall that to complete the proof of Theorem T we must demonstrate that, for a generic set of economies, there is no $\delta\in\Delta^{I+1}$ such that

$$\delta\cdot\Phi(x^*,p^*,\theta^*)=\delta\cdot\widetilde{\lambda}^*\cdot\psi(x^*)\ \square\ [D_p\hat{G}(p^*,\theta^*)]^{-1}\cdot \mathrm{A}(p^*)=[\underline{0}]^T.$$

So, let $\delta\in\Delta^{I+1}$ be such that $\delta_{i_0}>0$ for some $i_0\in\mathcal{I}$. Use the result in Proposition 6 to specify a set of $L+1$ agents, denoted $\{i_0,i_1,\ldots,i_L\}$, and a set of states $\widetilde{\mathcal{S}}$, such that $0\neq\delta_{i_0}\lambda^{i_0}_s{}^*\neq\delta_{i_m}\lambda^{i_m}_s{}^*$ for each $s\in\widetilde{\mathcal{S}}$ and each $m\in\{1,2,\ldots,L\}$. Use the specified set of agents and the set $\widetilde{\mathcal{S}}$ of states to construct the endowment perturbation specified above with $\{\gamma_{1s},\gamma_{2s},\ldots,\gamma_{Ls}\}$, $s\in\widetilde{\mathcal{S}}$, being arbitrary numbers. For each $s\in\widetilde{\mathcal{S}}$, apply Lemma L with $\delta_{i_m}\lambda^{i_m}_s{}^*$ playing the role of a_m, $m\in\{0,1,\ldots,L\}$, with $\{\gamma_{1s},\gamma_{2s},\ldots,\gamma_{Ls}\}$ playing the role of $\{\alpha_1,\alpha_2,\ldots,\alpha_L\}$, and with the set of vectors $\{V^{i_1}_s(p^*)-V^{i_0}_s(p^*),V^{i_2}_s(p^*)-V^{i_0}_s(p^*),\ldots,V^{i_L}_s(p^*)-V^{i_0}_s(p^*)\}$ playing the role of $\{v_1,\ldots,v_L\}$. The Lemma can be applied by invoking the spanning result obtained Proposition 3. It follows that any vector $\delta \cdot \widetilde{\lambda}^* \cdot \Delta\psi(x^*)$ with LS non-zero coordinates can be generated by suitably picking the set of numbers $\{\gamma_{1s}, \gamma_{2s}, \ldots, \gamma_{Ls}\}$ for each $s \in \widetilde{\mathcal{S}}$ since LS of its coordinates can be controlled independently.

The perturbation of endowments specified above also changes the matrix $D_p\hat{G}(p^*, \theta^*)$ which we now analyze. Consider a given state $s \in \widetilde{\mathcal{S}}$. For $i \in \mathcal{I}$, let $\Delta D_{p_s}[\hat{g}^i_s(p^*, {\theta^i}^*) - \hat{\omega}^i_s]$ and $\Delta D_{p_s}\hat{G}_s(p^*, \theta^*)$ denote the changes induced, respectively, in the matrices $D_{p_s}[\hat{g}^i_s(p^*, {\theta^i}^*) - \hat{\omega}^i_s]$ and $D_{p_s}\hat{G}_s(p^*, \theta^*)$, by the perturbation of endowments. The Slutsky decomposition of the matrix $D_{p_s}[\hat{g}^i_s(p^*, {\theta^i}^*) - \hat{\omega}^i_s]$ gives us[5]

$$D_{p_s}[\hat{g}^i_s(p^*, {\theta^i}^*) - \hat{\omega}^i_s] = {\lambda^i_s}^* K^i_s(p^*) - V^i_s(p^*) \cdot [\hat{g}^i_s(p^*, {\theta^i}^*) - \hat{\omega}^i_s]^T,$$

where $K^i_s(p^*)$ is a symmetric matrix of dimension $L \times L$. We note that ${\lambda^i_s}^*$, $K^i_s(p^*)$ and $V^i_s(p^*)$ for $i \in \mathcal{I}$ and $s \in \mathcal{S}$ are not affected by the specified perturbation of endowments since income, and hence demand, are not affected. Now, by making use of the induced changes to the excess demands of the agents, $\Delta\hat{z}^i_s$, and the fact that, for $s \in \mathcal{S}$, $\Delta D_{p_s}\hat{G}_s(p^*, \theta^*) = \sum_i \Delta D_{p_s}[\hat{g}^i_s(p^*, {\theta^i}^*) - \hat{\omega}^i_s]$, we obtain that

$$\Delta D_{p_s}\hat{G}_s(p^*, \theta^*) = -\sum_{m=0}^{L} V^{i_m}_s(p^*) \cdot \left[\Delta\hat{z}^{i_m}_s\right]^T =$$

$$-V^{i_0}_s \sum_{m=1}^{L} \gamma_{ms} \left[V^{i_m}_s(p^*) - V^{i_0}_s(p^*)\right]^T$$

$$+ \sum_{m=1}^{L} V^{i_m}_s \gamma_{ms} \left[V^{i_m}_s(p^*) - V^{i_0}_s(p^*)\right]^T =$$

$$\sum_{m=1}^{L} \gamma_{ms} \left[V^{i_m}_s(p^*) - V^{i_0}_s(p^*)\right] \cdot \left[V^{i_m}_s(p^*) - V^{i_0}_s(p^*)\right]^T.$$

To ease the notational burden, relabel each coordinate $[V^{i_m}_{ls}(p^*) - V^{i_0}_{ls}(p^*)]$ as $b^{i_m}_{ls}$ for each $m \in \{1, 2, \ldots, L\}$ and each $l \in \mathcal{L} \setminus \{0\}$. By writing out the product above, we obtain the matrix of dimension $L \times L$,

[5] See, e.g., Geanakoplos and Polemarchakis (1980).

$\Delta D_{p_s}\hat{G}_s(p^*, \theta^*) =$

$$\begin{bmatrix} \sum_{m=1}^{L} \gamma_{ms} b_{1s}^{i_m} b_{1s}^{i_m} & \sum_{m=1}^{L} \gamma_{ms} b_{1s}^{i_m} b_{2s}^{i_m} & \dots & \sum_{m=1}^{L} \gamma_{ms} b_{1s}^{i_m} b_{Ls}^{i_m} \\ \sum_{m=1}^{L} \gamma_{ms} b_{2s}^{i_m} b_{1s}^{i_m} & \sum_{m=1}^{L} \gamma_{ms} b_{2s}^{i_m} b_{2s}^{i_m} & \dots & \sum_{m=1}^{L} \gamma_{ms} b_{2s}^{i_m} b_{Ls}^{i_m} \\ \vdots & \vdots & & \vdots \\ \sum_{m=1}^{L} \gamma_{ms} b_{Ls}^{i_m} b_{1s}^{i_m} & \sum_{m=1}^{L} \gamma_{ms} b_{Ls}^{i_m} b_{2s}^{i_m} & \dots & \sum_{m=1}^{L} \gamma_{ms} b_{Ls}^{i_m} b_{Ls}^{i_m} \end{bmatrix} \quad (10)$$

which happens to be symmetric.

Let us now describe the perturbation to utilities, Δu. Consider an agent $i \in \mathcal{I}$, and construct Δu by placing a quadratic term, that we now describe, in the coordinate that corresponds to agent i, and by placing zeros in the other coordinates. This quadratic term is such that the linear term subsequently added to the vectors of first derivatives of u^i amounts to zero at the CE. Hence, it leaves aggregate demand unaffected, but changes the matrix of second derivatives of u^i.[6] Furthermore, this quadratic term induces, for each $s \in \mathcal{S}$, a change in the matrix $K_s^i(p^*)$ by the addition of a symmetric matrix that cancels out with the matrix in (10) above.

Since, from Assumption A.3, a variation of p_s only affects excess demand at state s, we have that the perturbation $(\Delta u, \Delta\omega)$ specified above is such that $[D_p\hat{G}(p^*, \theta^*)]^{-1}$ is not changed. Therefore, it generates the vector $\delta \cdot \widetilde{\lambda}^* \cdot \Delta\psi(x^*) \,\square\, [D_p\hat{G}(p^*, \theta^*)]^{-1}$ as desired for at least LS of its coordinates. Now, from the result stated in Proposition 4, any matrix obtained from $\mathrm{A}(p^*)$ by dropping the vectors that correspond to any state has at least $I+1$ linearly independent rows and thus we can choose the perturbation $(\Delta u, \Delta\omega)$ as to generate non-zero entries in those components of $\delta \cdot \widetilde{\lambda}^* \cdot \psi(x^*) \,\square\, [D_p\hat{G}(p^*, \theta^*)]^{-1}$ that correspond to some set of $I+1$ linearly independent rows from $\mathrm{A}(p^*)$. It follows that $\delta \cdot \widetilde{\lambda}^* \cdot \psi(x^*) \,\square\, [D_p\hat{G}(p^*, \theta^*)]^{-1} \cdot \mathrm{A}(p^*) \neq [\underline{0}]^T$ is guaranteed. Then, by applying a transversality argument, we obtain that $\delta \cdot \Phi(x^*, p^*, \theta^*) \neq [\underline{0}]^T$ for each $(u, \omega) \in \widetilde{\Gamma}$, where $\widetilde{\Gamma} \subset \widehat{\Gamma}$ is a generic set.

[6] It is known that by adding a suitable quadratic term to u^i, one can induce any perturbation of the matrix $K_s^i(p^*)$, for $i{\in}\mathcal{I}$ and $s{\in}\mathcal{S}$, by the addition of a symmetric matrix. See, e.g., Geanakoplos and Polemarchakis (1980).

Since δ was chosen arbitrarily, it follows that the matrix $\Phi(x^*, p^*, \theta^*)$ has rank $I+1$ for a generic set of economies $\widetilde{\Gamma}$. This completes the proof of Theorem T.

REMARK 5. The GP result holds for a generic set of economies. Of course, there are non-generic economies for which some CE are not CS. As in GP,consider an economy $(u, \omega) \in \Gamma$ for which there is a CE such that no agent trades any good at any state. Then, clearly, the last term in equation (3) amounts to zero and, therefore, the contribution to the change of utility of each agent due to the change in relative prices vanishes. So, given a reallocation of asset holdings $\mathrm{d}\theta$, $\mathrm{d}u(x^*)$ only captures the effect of a pure redistribution of income and, therefore, no improvement can be induced. However, we know that the economy (u, ω) belongs to a non-generic set since, by changing slightly the parameter ω, we move to a new economy such that some agents trade at each CE, which implies that the set that contains (u, ω) is not open.

REMARK 6. One would like to know whether the bound on the number of agents is tight. If $LS < I+1 \leq L(S+1)$, then the argument given to prove Theorem T fails to hold. To see this notice that, since the result obtained in Proposition 6 is in terms of ratios across states, one state must be dropped and used as a reference. Therefore, we are able only to control LS coordinates of the vector $\delta \cdot \widetilde{\lambda}^* \cdot \Delta\psi(x^*)$. Therefore, to show that the matrix $\Phi(x^*, p^*, \theta^*)$ has rank $I+1$, the set of vectors

$$\{r_0 \,\square\, [V^1(p^*) - V^0(p^*)], r_0 \,\square\, [V^I(p^*) - V^0(p^*)], r_1 \,\square\, [V^1(p^*) - V^0(p^*)]\}$$

needs to be linearly independent when considering any LS coordinates of them, which can be achieved only if $I+1 \leq LS$, a condition which is satisfied by imposing $I < LS$ as stated in the hypotheses of Theorem T.

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