A NOTE ON THE TWO-INPUT ARC ELASTICITY OF SUBSTITUTION*

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Resumen: En esta nota se propone una fórmula de la elasticidad arco de substitución para dos factores que surge naturalmente del concepto original de Hicks-Robinson y preserva sus características sobresalientes. En particular, la fórmula propuesta: (i) corresponde al valor medio de las elasticidades puntuales de substitución a lo largo del arco logarítmico de la relación del precios de los factores, y conduce por tanto a la exacta estimación de las funciones de producción CES, y (ii) su relación con la variación discreta de la participación de los factores en el producto total es formalmente análoga al supuesto de la elasticidad puntual.

Abstract: This note suggests a measure for the two-input arc elasticity of substitution that comes up naturally and preserves the salient characteristic of the Hicks-Robinson original concept. In particular, (i) it gives the average value of point substitution elasticities over the logarithmic arc of the input price ratio, and leads therefore to the exact estimation of the CES production function family, and (ii) its relationships with the discrete change in factor shares are formally parallel to those well known for the point elasticity assumption.

Clasificación JEL: D24, D33

Palabras clave: arc elasticity, substitution elasticity, elasticidad arco, elasticidad de substitución

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1. Introduction

This note presents a measure for the two-input arc elasticity of substitution that comes up naturally and preserves the salient characteristics of the Hicks-Robinson original concept. The measure might be of some interest in empirical studies where substitution possibilities are derived from the estimation of production function (or dual cost function) parameters. On the other hand, the topic is formally parallel to the demand elasticity case where alternative formulations for the arc elasticity of demand have been developed in the literature for a long time.[1] In section 2, a proposition demonstrates that it is the average value of point substitution elasticities over the logarithmic arc of the input price ratio, and leads therefore to the exact estimation of the well-known family of CES production functions. Section 3 shows that the relationships between the arc substitution elasticity and the discrete change in factor relative shares are formally parallel to those already known for the point substitution elasticity, the purpose for which the concept was originally introduced by Hicks (1932) and Robinson (1933). The analysis is confined to the two-input case because, as Hicks and Allen (1934) clearly pointed out in their pioneering lengthy article on the subject, when more than two inputs are involved, each substitution elasticity is represented by a directional derivative, an elusive concept. That is the case of the widely used Allen partial elasticity of substitution. A noteworthy exception, however, seems to be the generalization suggested by Morishima (1967), which has received a great deal of attention.[2]

Assuming (unnecessarily) linear homogeneity of the production function, Morishima defined the concept for any two inputs as the logarithmic derivative of the ratio of the inputs with respect to the logarithmic derivative of their marginal rate of substitution, with all other marginal rates of substitution kept unchanged. Now, however, under competitive equilibrium conditions, constant-output demand functions are homogeneous of zero degree in factor prices (independently of homogeneity in the production function), which means that allowing only one input price ratio to vary implies that the meaningful variation is entirely attributed to changes in only one price. This is the reason why, as noted by Morishima (1967) himself, his

[1] For details see Vázquez (1995, 1998).

[2] See Anderson and Moroney (1993), Blackorby and Russell (1981, 1989), Kang and Brown (1981), Koizumi (1976), Kuga (1979), Kuga and Murota (1972) and Murota (1977).

generalisation may be written as the difference between the cross and the own constant-output price demand elasticities,[3] and thus nothing substantial is won for the purpose of this note, as it is just confined to the two-input case.

2. The Basic Model

Let $Y = F(L, K)$ be any neo-classical twice-differentiable production function, with positive and diminishing marginal products over the relevant range of inputs, where Y is output, and L and K are labour and capital inputs. The point elasticity of substitution, denoted σ, is defined for a given output (Lerner, 1933:68) by:

$$\sigma = \frac{\mathrm{dln}\,(\mathrm{L}/\mathrm{K})}{\mathrm{dln}\,(\mathrm{F_K}/\mathrm{F_L})} \tag{1}$$

where F_K and F_L are the marginal product of capital and labour, respectively. Now, under competitive equilibrium conditions, (1) becomes:

$$\sigma = \frac{\mathrm{dln}\,(\mathrm{L}/\mathrm{K})}{\mathrm{dln}\,(\mathrm{r}/\mathrm{w})} \tag{2}$$

where r is the rental value of capital and w the wage rate.[4]

Next let $P_0(L_0, K_0)$ and $P_1(L_1, K_1)$ be any two points on the economic region of factor space, and let the subscripts 0 and 1 denote the price values at these points. Thus, letting σ^A be the arc price elasticity of substitution, the following definition comes naturally from (2), by simply replacing the differentials for finite changes:

$$\sigma^{\mathrm{A}} = \frac{\ln(\mathrm{L_0}/\mathrm{K_0}) - \ln(\mathrm{L_1}/\mathrm{K_1})}{\ln(\mathrm{r_0}/\mathrm{w_0}) - \ln(\mathrm{r_1}/\mathrm{w_1})} = \frac{\ln(\mathrm{L_0}/\mathrm{L_1}) - \ln(\mathrm{K_0}/\mathrm{K_1})}{\ln(\mathrm{w_1}/\mathrm{w_0}) - \ln(\mathrm{r_1}/\mathrm{r_0})}. \tag{3}$$

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[3] Incidentally, it has been perhaps insufficiently realized that Morishima generalization also holds under long-run equilibrium conditions.

[4] It should be noted, however, that definition (2) also holds under imperfect equilibrium conditions, provided that (*e.g.* Vázquez, 1969:564-566)

$$1{+}1/\eta_L{=}k\big(1{+}1/\eta_K\big),$$

where k is a positive constant, and η_L and η_K are, respectively, the supply elasticity for labour and capital.

Note that σ^A is independent of the measurement units of inputs and prices, symmetric with respect to both factors and arc ending points, varying from zero, when the isoquants are right-angled, to positive infinity, when they are linear. Note also that σ^A remains constant, when inputs change proportionally, and that $\sigma^{\mathrm{A}} \to \sigma$, when $(\mathrm{L}_0,\mathrm{K}_0) \to (\mathrm{L}_1,\mathrm{K}_1)$.

We now present the following proposition:

PROPOSITION. *Given any two points in the economic region of the factor space, there is one point in the interval such that its elasticity of substitution is equal to the arc elasticity of substitution defined by (3).*

PROOF. For a given level of output and perfect competition, the following substitution function can be written:

$$\frac{\mathrm{L}}{\mathrm{K}} = \mathrm{f}\left(\frac{\mathrm{r}}{\mathrm{w}}\right) \tag{4}$$

If the isoquant is strictly convex to the origin, then (4) is an increasing function. Letting $u = \ln(L/K)$ and $v = \ln(r/w)$, (4) can be written as

$$\mathrm{u} = \varphi(\mathrm{v}) \tag{5}$$

and hence, by definition,

$$\sigma = \varphi'(\mathrm{v}) \tag{6}$$

If (v_0, v_1) is the interval on the logarithmic substitution function which corresponds to points $P_0(L_0, K_0)$ and $P_1(L_1, K_1)$ on the isoquant, then the mean value theorem allows us to write:

$$\varphi'\left(\mathrm{v_S}\right) = \frac{\varphi\left(\mathrm{v}_0\right) - \varphi\left(\mathrm{v}_1\right)}{\mathrm{v}_0 - \mathrm{v}_1} = \sigma\left(\mathrm{v_S}\right) \qquad \left(\mathrm{v}_1 \leq \mathrm{v_S} \leq \mathrm{v}_0\right). \tag{7}$$

Through the inverse transformation we obtain formula (3) above. ■

On the other hand, from (3), (6) and (7), it follows that

$$\sigma^{\mathrm{A}} = \frac{\int_{\mathrm{v}_1}^{\mathrm{v}_0} \varphi'(\mathrm{v})\mathrm{dv}}{\int_{\mathrm{v}_1}^{\mathrm{v}_0} \mathrm{dv}} = \frac{\int_{\mathrm{v}_1}^{\mathrm{v}_0} \sigma(\mathrm{v})\mathrm{dv}}{\int_{\mathrm{v}_1}^{\mathrm{v}_0} \mathrm{dv}} \tag{8}$$

which expresses literally that definition (3) is equal to the average value of point substitution elasticities over the logarithmic arc of the input price ratio. Alternatively, arc substitution elasticity equals the slope of the chord connecting two points on the logarithmic graph of the substitution function transform. Obviously, if the point substitution elasticity is constant along the isoquant, i.e., if the production function is of the CES type, then $\sigma^A = \sigma$, a constant. Conversely, for the arc substitution elasticity to be constant, the production function must belong to the CES family.[5]

REMARK. (3) defines the unique measure of arc elasticity that satisfies the consistency requirement that whenever the point elasticity of substitution is constant, the arc elasticity of substitution also has the same constant value, irrespective of the span of the arc, and vice-versa.

On the other hand, to show that arc substitution elasticity is really ordinary elasticity with an index number problem thrown in, note that definition (3) relies on the properties of the so-called *logarithmic mean*, independently introduced by Vartia (1974) and Sato (1976) for the purpose of deriving two termed ideal long-change index numbers.[6] They defined the logarithmic mean $\psi(x_0, x_1)$ for nonnegative x_0 and x_1 by:

$$\begin{aligned} \psi\,(\mathrm{x}_0,\mathrm{x})_1 &= \frac{\mathrm{x}_1 - \mathrm{x}_0}{\ln \mathrm{x}_1 - \ln \mathrm{x}_0} && \text{for} \quad \mathrm{x}_0 \neq \mathrm{x}_1 \\ &= \mathrm{x}_0 && \text{for} \quad \mathrm{x}_0 = \mathrm{x}_1 \end{aligned}$$

Now, it has been shown by Vartia (1976) and Sato (1976) that the logarithmic mean is really a mean with the normal properties that weight functions are expected to have. Then, since

$$\ln\,(\mathrm{x}_1/\mathrm{x}_0) = \frac{\mathrm{x}_1 - \mathrm{x}_0}{\psi\,(\mathrm{x}_0, \mathrm{x}_1)}$$

[5] As it is known, the assumption of constancy in the elasticity of substitution determines only the isoquant equation, but not the analytic production function. Together with homogeneity (homotheticity), however, this assumption implies a CES production function, which has been derived from different alternative approaches. For details and references, see Vázquez (1969, 1971).

[6] To be sure, both indices were presented in Vartia (1974). Sato (1976) documents an independent rediscovery of Vartia index II. Vartia, in turn, documents an independent rediscovery of his index I, first derived from in a rather cumbersome mathematical form by Montgomery (1937:30-39).

we see that σ^A can be interpreted as a relative change in variables with respect to the logarithmic mean, or, simply, as an ordinary elasticity with an index number problem thrown in.[7]

3. Relationships Between Arc Substitution Elasticity and Factor Relative Shares

To derive the relationships between the arc elasticity of substitution and the discrete change in factor relative shares, we use now the superscripts 0 and 1 to denote respectively the relative shares of labour (S_L) and capital (S_K) in total cost, corresponding to points P_0 and P_1. Therefore

$$\frac{\mathrm{S_L^0}}{\mathrm{S_K^0}} = \frac{\mathrm{w_0 L_0}}{\mathrm{r_0 K_0}} \qquad \left(\mathrm{S_L^0} + \mathrm{S_K^0} = 1\right) \tag{9}$$

$$\frac{\mathrm{S_L^1}}{\mathrm{S_K^1}} = \frac{\mathrm{w_1 L_1}}{\mathrm{r_1 K_1}} \qquad \left(\mathrm{S_L^1} + \mathrm{S_K^1} = 1\right) \tag{10}$$

By taking logarithms in (9) and (10), and using (3), we obtain the following well-known Hicks-Robinson parallel relationships:[8]

$$\frac{\ln\left(\mathrm{S_L^0/S_K^0}\right) - \ln\left(\mathrm{S_L^1/S_K^1}\right)}{\ln\left(\mathrm{w_0/r_0}\right) - \ln\left(\mathrm{w_1/r_1}\right)} = 1 - \sigma^{\mathrm{A}} \tag{11}$$

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[7] Even though the topic is outside the scope of this note, note that this immediately suggests the possibility of extending formula (3), as well as relations (11) and (12) derived below, to the case where variables are aggregates and indices 0 and 1 two time periods, provided that price (quantum) index be constructed using the logarithmic mean as the weight function. Since the Vartia-Sato indices satisfy this condition, they come *ad hoc*. Unfortunately, however, Diewert (1978) has showed that Vartia index I is exact only for the Cobb-Douglas aggregator function, whereas Sato (1976) showed that his and Vartia index II is exact only for the CES aggregator function. These results also follow rather tersely from the remark.

[8] See *e.g.* Kahn (1933), Solow (1958), Kravis (1959), Bronfenbrenner (1960) and Vázquez (1991). For the analysis in a multifactor multisector model, see Bigman (1978), and for a discussion on factor shares and different substitution elasticity concepts in a multifactor model, see Samuelson (1973) and Sato and Koizumi (1973a, 1973b).

$$\frac{\ln\left(\mathrm{S}_{\mathrm{L}}^{0}/\mathrm{S}_{\mathrm{K}}^{0}\right)-\ln\left(\mathrm{S}_{\mathrm{L}}^{1}/\mathrm{S}_{\mathrm{K}}^{1}\right)}{\ln\left(\mathrm{L}_{0}/\mathrm{K}_{0}\right)-\ln\left(\mathrm{L}_{1}/\mathrm{K}_{1}\right)}=1-\frac{1}{\sigma^{\mathrm{A}}} \tag{12}$$

To analyse relations above, suppose first that $w_0/r_0 > w_1/r_1$, expressing that labour price relative to that of capital decreases from point P_0 to point P_1 (for $w_0/r_0 < w_1/r_1$ the argument would be reversed). Then, since the denominator in the left hand side of (11) is positive in sign, the numerator will be positive, null, or negative depending on whether the arc elasticity of substitution is less than, equal to, or greater than unity. That is,

$$\frac{\mathrm{S}_{\mathrm{L}}^{0}}{\mathrm{S}_{\mathrm{K}}^{0}} \underset{<}{\geq} \frac{\mathrm{S}_{\mathrm{L}}^{1}}{\mathrm{S}_{\mathrm{K}}^{1}}, \text{ depending on whether } \sigma^{\mathrm{A}} \underset{<}{\geq} 1. \tag{13}$$

Since total relative shares sum unity, we also have that

$$\mathrm{S}_{\mathrm{L}}^{0} \underset{<}{\geq} \mathrm{S}_{\mathrm{L}}^{1}, \text{ and } \mathrm{S}_{\mathrm{K}}^{0} \underset{>}{\leq} \mathrm{S}_{\mathrm{K}}^{1}, \text{ depending on whether } \sigma^{\mathrm{A}} \underset{<}{\geq} 1. \tag{14}$$

Next consider the effect of a discrete change in the employment of a factor on its relative shares. Suppose now that $L_0/K_0 < L_1/K_1$, meaning that the quantity of labour relative to the quantity of capital rises from P_0 to P_1. Then, it is obvious that

$$\frac{\mathrm{S}_{\mathrm{L}}^{0}}{\mathrm{S}_{\mathrm{K}}^{0}} \underset{>}{\leq} \frac{\mathrm{S}_{\mathrm{L}}^{1}}{\mathrm{S}_{\mathrm{K}}^{1}}, \text{ depending on whether } \sigma^{\mathrm{A}} \underset{<}{\geq} 1. \tag{15}$$

Recalling that relative shares add up to unity, we have

$$\mathrm{S}_{\mathrm{L}}^{0} \underset{>}{\leq} \mathrm{S}_{\mathrm{L}}^{1}, \text{ and } \mathrm{S}_{\mathrm{K}}^{0} \underset{<}{\geq} \mathrm{S}_{\mathrm{K}}^{1}, \text{ depending on whether } \sigma^{\mathrm{A}} \underset{<}{\geq} 1,$$

which can be rewritten as

$$\mathrm{S}_{\mathrm{L}}^{0} \underset{<}{\geq} \mathrm{S}_{\mathrm{L}}^{1}, \text{ and } \mathrm{S}_{\mathrm{K}}^{0} \underset{>}{\leq} \mathrm{S}_{\mathrm{K}}^{1}, \text{ depending on whether } \sigma^{\mathrm{A}} \underset{>}{\leq} 1. \tag{16}$$

Quite clearly, the propositions that follow from (13)-(16) are entirely analogous to those of the point substitution elasticity, and they not need therefore be repeated here.

4. Conclusions

In this note, we have described a simple measure for the arc elasticity of substitution that wells up naturally from the point substitution elasticity formula and preserves its salient characteristics. An essential feature is that its relationships with the discrete changes in factor relative shares are formally identical to those well known for the point elasticity case. It should be stressed, however, that these relationships, whether they concern point or arc elasticities, are necessarily true by definition and therefore merely tautological, and therefore empty of empirical substance. Of course, this by no means signifies that they are useless. On the contrary, being identities, they allow us to obtain the same qualitative and quantitative information about factor shares from the knowledge of the substitution elasticity and vice-versa, and this is important in those cases where data on prices or quantities are not easy to come by, or when no clear guidelines for choosing the appropriate production function form for estimation exist.

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