ON CONSTANT ELASTICITIES OF DEMAND

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Resumen: Mientras la matriz de Slutsky y la teoría de la dualidad se han utilizado para demostrar que las funciones de demanda con elasticidades constantes implican: elasticidades de renta unitarias, elasticidades cruzadas de la demanda nulas y elasticidades directas iguales a menos uno; esta nota demuestra que estos resultados se pueden obtener también directamente suponiendo simplemente que se verifica la ecuación de balance con estricta igualdad.

Abstract: While the Slutsky matrix and duality theory have been used to establish that constant elasticity demand functions imply unitary income elasticities, zero cross-price elasticities and own-price elasticities equal to minus one, this note shows that these results can also be strightforwardly derived from the simple assumption that demand functions satisfy the budget constraint with strict equality.

1. Introduction

The easy simplicity of the elasticity concept and the fact that elasticities are pure numbers have led economists to see their estimation as a primary aim of empirical studies. In econometric studies of consumer demand it has long been a common practice to assume that demand are elasticities constant. While this simple assumption facilitates the parametric estimation of demand elasticities (since it means that the underlying demand functions must be log-linear), it also imposes restrictions upon the magnitude of the elasticity values.[1] Slutsky equations, as well as duality

[1] Pareto (1911, pp. 613-616) was early in pointing out that the Marshallian constant demand curve is inadmissible, except in the special case when elasticity is unity.

theory, have been used to fully describe the framework within which such demand funtions are consistent with the theoretical restrictions implied by the theory of consumer choice. In particular, it turns out that income elasticities will be unitary, the own-price elasticities will be equal to −1, and the cross-price elasticities will all equal to zero, see, e.g., Basmann *et al.* (1973), Dimasi and Schap (1985), Samuelson (1965), and Willig (1976). The purpose of this note is to show that these results can also be derived by simply assuming that demand functions satisfy the budget constraint with strict equality.

2. Implications of Constant Elasticities of Demand

Let the standard neoclassical twice-differentiable demand functions be $x_i = x_i(\mathbf{p}, y)$, where x_i denotes the quantity demanded of the ith good ($i = 1, \ldots, n$), $\mathbf{p} = (p_1, \ldots, p_n)$ is a vector representing the prices, and y is the consumer's income. Then,

$$\sum_{i=1}^{n} x_i p_i = y, \tag{1}$$

and

$$\sum_{i=1}^{n} S_i = 1, \tag{2}$$

where S_i stands for the relative share of the ith good in total income (expenditure), i. e., $S_i = x_i p_i / y$ ($i = 1, \ldots, n$).

Next, denote the elasticity of the ith good with respect to price p_j by η_{ij} and with respect to income by η_{iy}. Then, since demand functions are homogeneous of degree zero in prices and income, Euler's theorem allows us to write the aggregation condition

$$\sum_{j=1}^{n} \eta_{ij} = -\eta_{iy}, \quad (i = 1, \ldots, n) \tag{3}$$

first derived by Slutsky (1915), and then by Hicks (1937) and Schultz (1938) from the second order conditions for utility maximization. Notice that (3) establishes

PROPOSITION 1. *If all cross-price elasticities of demand of a commodity vanish, then its own-price demand elasticity is numerically equal to its income elasticity.*

On the other hand, partial differentiation of (1) with respect to p_j yields, after some manipulations, Cournots aggregation condition

$$\sum_{i=1}^{n} S_i\eta_{ij} = -S_j, \quad (j = 1, \dots, n) \tag{4}$$

which can also be written as

$$\sum_{i \neq j}^{n} S_i\eta_{ij} = -S_j(\eta_{jj} + 1), \quad (j = 1, \dots, n)$$

from which we have,

PROPOSITION 2. *If the cross-demand elasticities of all goods with respect to a given price pj are all zero, then the own-price demand elasticity of the jth good must be* -1.

Finally, by partially differentiating (1) with respect to y, we get the Engel aggregation condition[2]

$$\sum_{i=1}^{n} S_i\eta_{iy} = 1. \tag{5}$$

We now present the following theorem:

THEOREM 1. *If the price elasticities of demand for a good are all constant, then the cross-price demand elasticities all vanish and the own-price demand elasticity equals minus one.*

PROOF. By assuming that η_{ij} constant, and noting that $\partial S_i / \partial p_j = S_i / p_j(\eta_{ij} + \delta_{ij})$, where δ_{ij} is the Kronecker symbol (i. e., $\delta_{ij} = 1$, if $i = j$, and $\delta_{ij} = 0$, if $i \neq j$), partial differentiation of (4) with respect to p_j yields

[2] Notice that (2), (3) and (4) imply (5).

$$\sum_{i \neq 1}^{n} S_i(\eta_{ij})^2 + S_j(\eta_{jj} + 1)^2 = 0, \quad (j = 1, \ldots, n) \tag{6}$$

which, for $S_i > 0$, implies $\eta_{jj} = -1$, and $\eta_{ij} = 0$ $(i \neq j)$. ■

COROLLARY 1. *If the price demand elasticities are constant, then the income demand elasticities become unity.*

PROOF. This follows immediately from theorem 1 and proposition 1.

THEOREM 2. *If the income elasticities are all constant, then they must all be equal to 1.*

PROOF. Using (2), (5) can be rewritten as

$$\sum_{i=1}^{n} S_i(\eta_{iy} - 1) = 0,$$

whose partial differentiation with respect to y, assuming η_{iy} = constant, and noting that,[3] $\partial S_i / \partial y = S_i / y(\eta_{iy} - 1)$, gives

$$\sum_{i=1}^{n} S_i(\eta_{iy} - 1)^2 = 0, \tag{7}$$

which implies $\eta_{iy} = 1$ $(i = 1, \ldots, n)$ ■

THEOREM 3. *If the price demand elasticity of a good is independent of income, then the good's income elasticity is also independent of price, and vice-versa.*

PROOF. Since

$$\frac{\partial \eta_{ij}}{\partial y} = \frac{p_j \partial^2 x_i}{x_i \partial p_j \partial y} - \frac{1}{y}(\eta_{ij}\, \eta_{iy})$$

and

[3] Incidentally, this derivative indicates the tautology of Engel's law, according to which the proportion of income devoted to necessary goods $(0 < \eta_{iy} < 1)$ decreases as income increases, while the opposite holds for luxury goods $(\eta_{iy} > 1)$. See, for example, Derycke (1964, pp. 129-130), and Wold and Juréen (1953, pp. 323).

$$\frac{\partial \eta_{iy}}{\partial p_j} = \frac{y\partial^2 x_i}{x_i \partial y \partial p_j} - \frac{1}{p_j}(\eta_{ij}\,\eta_{iy})$$

we have the duality relation

$$\frac{\partial \eta_{ij}}{\partial \ln y} = \frac{\partial \eta_{iy}}{\partial \mathrm{in} p_j},$$

which establishes that

$$\frac{\partial \eta_{ij}}{\partial y} = 0 \Leftrightarrow \frac{\partial \eta_{iy}}{\partial p_j} = 0 \quad (i, j = 1, 2, \dots, n)$$ ■

COROLLARY 2. *(i) If the price demand elasticities of a good are all independent of the income, then the good's income elasticity is constant. (ii) If this holds for all goods, then income elasticities are all unity.*

PROOF. (i) is immediate from theorem 3 and the aggregation condition (3); (ii) then follows directly from theorem 2.

3. Conclusions

In econometric studies of consumer demand it has long been postulated that demand elasticities are constant. This simple assumption facilitates the estimation of demand elasticities in parametric form because, as first noted by Moore (1926), demand functions must then be of the form[4]

$$x_i = A_i\, y^{\eta_{iy}} \prod_{j=1}^{n} p_j^{\eta_{ij}}, \quad (i, j = 1, 2, \dots, n) \tag{8}$$

and taking logarithms leads to the log linear representation:

$$\ln x_i = \ln A_i + \eta_{iy} \ln y + \eta_{ij} \sum_{j=1}^{n} \ln p_j \quad (i = 1, 2, \dots, n)$$

However the assumption of constant demand elasticities also imposes severe restrictions upon the own elasticity values. In this note we have shown that without resorting to duality theory or the integrability

[4] A similar specification was used in Wold and Juréen (1953, pp. 3, 105).

conditions, and by simply assuming that demand equations satisfy the budget constraint with strict equality, $\eta_{iy} = 1$, $\eta_{ii} = -1$, and $\eta_{ij} = 0$ for $i \neq j$. Substitution of these results into (8) gives us the correct form of the underlying demand function

$$x_i = A_i \, y p_i^{-1} \tag{9}$$

or, arranging, $x_i p_i / y = A_i$, where $A_i =$ constant, $0 < A_i < 1$, and $\sum_{i=1}^{n} = 1$. Notice that (9) simply says that the expenditure on each commodity is a constant fraction of the budget. With this fact in mind, the potential gains in estimation from choosing constant elasticities in empirieal demand works must then be balanced against the rather unattractive structure that this assumption imposes on the underlying demand function. Of course, the same conclusions apply input demand functions in the theory of production under the formally parallel assumption of cost minimization.[5]

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[5] For details, see Vázquez (1972), and Vázquez and Puu (1973).

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