SOME REMARKS ABOUT UNIQUENESS OF EQUILIBRIUM FOR INFINITE DIMENSIONAL ECONOMIES

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Resumen: En la teoría del equilibrio general existe una importante cantidad de trabajos destinados a encontrar condiciones que garanticen la unicidad del equilibrio walrasiano. La gran mayoría de ellos parte de la función exceso de demanda e impone condiciones para la existencia de un único cero para esta función, esto es, un único precio de equilibrio. Pero este procedimiento es muy restrictivo cuando los espacios de consumo dejan de ser subconjuntos de espacios de dimensión finita, donde la existencia de la función demanda no surge necesariamente como resultado de un proceso de maximización. Este trabajo en su primera parte, trata precisamente de uno de estos casos, y en su segunda parte generaliza a economías con infinitos bienes la condición de Mitjushim-Polterovich.

Abstract: For economies with infinitely many goods, with two different approaches we obtain sufficient conditions for uniqueness of competitive equilibrium. In the second approach we prove that the Mitjushim-Polterovich condition is a sufficient condition for uniqueness of equilibrium when the consumption space is a positive cone included in a Banach space. We do not suppose separability of the utility function.

Introduction

Conditions for uniqueness of equilibrium on economies with finite dimensional consumption spaces are well known, while uniqueness

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result in economies with infinite dimensional consumption spaces are scare.

Dana (1993) was the first one to obtain an uniqueness result in infinite dimensional consumption spaces. She considers the case of a pure exchange economy where the agent's consumption space is $L^p_+(\mu)$ and agents have additively separable utilities.

In the first approach, from the excess utility function we obtain an uniqueness result of competitive equilibrium when the consumption space is a measurable function space and the utility functions are separable (we do not assume the existence of demand function). Recall that when the consumption spaces are infinite dimensional vector spaces, the existence of demand function is not a necessary consequence of a maximization process. While this is not a serious obstacle for the study of existence of equilibrium, it is a serious one for the knowledge of the topological properties of the equilibrium set.

In our first approach we show that the excess utility function is a powerful tool for knowing the topological structure of the equilibrium set for the infinite dimensional case. Using the excess utility function we prove that economies with separable utilities and allocations defined on a probability space (the set of states can be infinite; is in this sense in that the paper considers an infinite number of goods) have local uniqueness, an odd number of equilibria, and we obtain sufficient conditions for global uniqueness.

Invoking the Pareto optimality of Walrasian equilibrium, we obtain a one to one correspondence between the solutions of the equation $e(\lambda) = 0$, and the equilibrium price, where e is the excess utility function. In our case, price is a measurable fuction in a probability space. In this way an infinite dimensional problem is reduced to a finite dimensional one.

In the second approach we generalize to the infinite dimensional case the Mitjushim-Polterovich condition. In this approach we assume that the excess utility function exists and we prove that the Mitjushim-Polterovich condition is a sufficient condition for uniqueness of the equilibrium in the infinite dimensional case.

The outline of the paper is the following:

We consider a pure exchange economy with consumption spaces that are a finite Cartesian product of measurable functions. Utility functions are additively separable (see section 1).

The properties of the excess utility function (see section 2) allow us to apply degree theory to the considered economies. We prove that the cardinality of the equilibrium set for these economies is odd and we obtain sufficient conditions for uniqueness of equilibrium to hold.

In section 3, by means of the excess utility function, we transform an infinite dimensional optimization problem in a finite dimensional one, obtaining a sufficient condition for uniqueness of equilibrium.

In section 4 we illustrate the developed theory with some examples.

In section 5 the proofs of theorems are given.

Finally, assuming that the excess demand function exists we prove that the Mitjushim-Polterovich condition can be applied for distributive economies with consumption spaces in a positive cone included in Banach spaces. Agents need not have additively separable utilities.

1. The Model

Let us consider a pure exchange economy with n agents and l goods at each state of the nature. The set of states is a measure space: (Ω, A, ν).

We assume that each agent has the same consumption space, $M = \Pi_{i=1}^{l} M_i$ where M_i is the space of all positive measurable functions defined on (Ω, A, ν).

Let $R_{++}^{l} = \{x \in R^l$ with all components positive$\}$.

Following Mas-Colell (1991), we consider the space A of the C^2 utility functions on R_{++}^{l}, strictly monotone, differentiably strictly concave and proper.

DEFINITION 1. *A C^2 utility function u is differentiably strictly convex, if it is strictly convex and every point is regular; that is the Gaussian curvature, C_x of each level surface of u, is a non null function in each x.*

For $x, y \in R^l$ we will write $x > y$ if $x_i \geq y_i$ $i = 1 \ldots l$ and $x \neq y$.

DEFINITION 2. *A utility function is strictly monotone if $x > y \Rightarrow u(x) > u(y)$.*

DEFINITION 3. *We say that $u \in C^2$ is proper if the limit of $|u'(x)|$ is infinite, when x approaches to the boundary of R_{++}^{l}, i.e. the set $B = \{x : x_i = 0$ for some $i = 1, \ldots, n\}$.*

We will consider the space $\boldsymbol{U}$ of all measurable functions $U : \Omega \times R^l_{++} \to R$, such that $U(s, \cdot) \in \Lambda$ for each $s \in \Omega$.

We introduce the uniform convergence in this space: $U_n \to U$ if $\|U_n - U\|_K \to 0$ for any compact $K \subset R^l_{++}$, where $\|U_n - U\|_K =$

$$ess \sup_{s \in \Omega} \max_{z \in K} \{|U_n(s, z) - U(s, z)| + |\partial U_n(s, z) - \partial U(s, z)|$$
$$+ |\partial^2 U_n(s, z) - \partial^2 U(s, z)|\}.$$

We say that a real number M, is the essential supremum of f, and we write, $ess \sup_{s \in \Omega} f(s)$ if $|f| \leq M$ for almost all $s \in \Omega$.

Each agent is characterized by his utility function u_i and by his endowment $w_i \in \boldsymbol{M}$.

From now on we will work with economies with the following characteristics:

a) The utility functions $u_i : \boldsymbol{M} \to R$ are separable. This means that they can be represented by

$$u_i(x) = \int_\Omega U_i(s, x(s)) dv(s) \quad i = 1, \ldots, n \tag{1}$$

where $U_i : \Omega \times R^l_{++} \to R$ and $U_i(s, \cdot)$ is the utility function at every state $s \in \Omega$.

b) The utility functions $U_i(s, \cdot)$ belongs to a fixed compact subset of Λ, for each $s \in \Omega$ and $U_i \in \boldsymbol{U}$.

c) The agents' endowments $w_i \in \boldsymbol{M}$ are bounded from above and bounded away from zero in every component, i.e. there exists h and H with $h < w_{ij}(s) < H$ for each $j = 1, \ldots, l$, and $s \in \Omega$.

The following definitions are standard.

DEFINITION 4. *An allocation of commodities is a list* $x = (x_1, \ldots, x_n)$ *where* $x : \Omega \to R^{ln}$ and $\sum_{k=1}^{n} x_k(s) \leq \sum_{k=1}^{n} w_k(s)$.

DEFINITION 5. *A commodity price system is a measurable function* $p : \Omega \to R^l_{++}$, *and for any* z R^l *we denote by* $\langle p, z \rangle$ *the real number* $\int_\Omega p(s)z(s)dv(s)$. *(We are not using any specific symbol for the Euclidean inner product in* R^l.*)*

DEFINITION 6. *The pair (p, x) is an equilibrium if:*

i) p is a commodity price system and x is an allocation,

ii) $\langle p, x_i \rangle \leq \langle p, w_i \rangle < \infty$ for all $i \in \{1, \dots, n\}$

iii) if $\langle p, z \rangle \leq \langle p, w_i \rangle$ with $z : \Omega \to R^l_{++}$, then

$$\int_\Omega U_i(s, x_i(s))dv(s) \geq \int_\Omega U_i(s, z(s))dv(s) \text{ for all } i \in \{1, \dots, n\}.$$

2. The Excess Utility Function

In order to obtain our results we introduce the excess utility function.

We begin by writing the following well known proposition:

PROPOSITION 1. *For each λ in the $(n-1)$ dimensional open simplex, $\Delta^{n-1} = \{\lambda \in R^n_{++}; \sum \lambda_i = 1\}$, there exists*

$$\bar{x}(\lambda) = \{\bar{x}_1(\lambda), \dots, \bar{x}_n(\lambda)\} \in R^{\ln}_{++}$$

solution of the following problem:

$$\max_{x \in R^{ln}} \sum_i \lambda_i U_i(x_i)$$
$$\textit{subject to } \sum_i x_i \leq \sum_i w_i \textit{ and } x_i \geq 0. \tag{2}$$

If U_i depend also on $s \in \Omega$, and $U_i(s, \cdot) \in \Lambda$ for each $s \in \Omega$, and $\lambda \in \Delta^{n-1}$, there exists $\bar{x}(s, \lambda) = \bar{x}_1(s, \lambda), \dots, \bar{x}_n(s, \lambda)$ solution of the following problem:

$$\max_{x(s) \in R^{ln}} \sum_i \lambda_i U_i(s, x_i(s))$$
$$\text{subject to } \sum_i x_i(s) \leq \sum_i w_i(s) \text{ and } x_i(s) \geq 0 \tag{3}$$

If $\gamma^j(s, \lambda)$ are the Lagrange multipliers of problem (3), $j \in \{1, \dots, l\}$, then from the first order conditions we have that

$$\lambda_i \frac{\partial U_i(s, \bar{x}_i(s, \lambda))}{\partial x^j} = \gamma^j(s, \lambda) \text{ with } i \in \{1, \dots, n\} \text{ and } j \in \{1, \dots, l\}.$$

Then the following identities hold

$$\lambda_i \partial U_i(s, \bar{x}_i(s, \lambda)) = \gamma(s, \lambda) \text{ for all } i = 1, \ldots, n; \text{ for all } s \in \Omega. \quad (4)$$

REMARK 1. From the Inada condition of "infinite marginal utility" at zero (definition 3), the solution of (3) must be strictly positive almost everywhere. Since $U(s, \cdot)$ is a monotone function, we can deduce that $\sum_{i=1}^{n} \bar{x}_i(s) = \sum_{i=1}^{n} w_i(s)$.

Let us now define the excess utility function.

DEFINITION 7. *Let $x_i(s, \lambda)$; $i \in \{1, \ldots, n\}$ be a solution of (3).*
We say that $e : \Delta^{n-1} \rightarrow R^n e(\lambda) = (e_1(\lambda), \ldots, e_n(\lambda))$, with

$$e_i(\lambda) = \frac{1}{\lambda_i} \int_\Omega \gamma(s, \lambda)[x_i(s, \lambda) - w_i(s)]dv(s), \; i = 1, \ldots, n. \quad (5)$$

is the excess utility function.

REMARK 2. Since the solution of (3) is homogeneous of degree zero, i.e. $\bar{x}(s, \lambda) = \bar{x}(s, \alpha\lambda)$ for any $\alpha > 0$, then we can consider e_i defined all over R^n_{++} by $e_i(\alpha\lambda) = e_i(\lambda)$ for all $\lambda \in \Delta^{n-1}_{++}$, $\alpha > 0$.

3. Equilibrium and the Excess Utility Function

Let us now consider the following problem:

$$\max_{x \in M} \sum_i \bar{\lambda}_i \int_\Omega U_i(s, x_i(s))dv(s)$$

$$\text{subject to } \sum_i x_i(s) \leq \sum_i w_i(s) \text{ and } x_i(s) \geq 0. \quad (6)$$

It is a well known proposition that an allocation $\bar{x}$ is Pareto optimal if and only if we can choose a $\bar{\lambda}$ such that $\bar{x}$ solves the above problem, with $\lambda = \bar{\lambda}$. Moreover, since a consumer with zero social weight receive nothing of value at a solution of this problem, we have that if $\bar{x}$ is a strictly positive allocation, that is $\{\bar{x} \in R^l_{++}\}$, all consumer has a positive social weight. See for instance Kehoe (1991). Reciprocally if $\bar{\lambda}$ is in the interior of the simplex, then from remark (1) the solution $x(., \lambda)$ of (6) is a strictly positive Pareto optimal allocation. (This is guaranteed also by the following boundary condition on preferences: $\{v(s) \in R^l_{++} : v(s) \succeq_i w_i(s)\}$ is closed for almost every state (*a.e.s.*), for all i and $w_i(s)$ strictly positive).

REMARK 3. Our approach requires that a Pareto optimal allocation exists. Observe that in our model the closedness condition introduced by Mas-Colell (1986) is satisfied. That is $\boldsymbol{U} = \{(u_1(x_1), \ldots, u_n(x_n)) : (x_1, \ldots, x_n)$ is a feasible allocation$\}$ is a closed subset of R^n_+.

From the first welfare theorem, we have that every equilibrium allocation is Pareto optimal.

Let $\overline{x}$ be an equilibrium allocation, then there exists a $\overline{\lambda}$ such that $\overline{x} = \{\overline{x}_1, \ldots, \overline{x}_n\} : \Omega \to R^n$ is a solution for the problem in the beginning of this section.

In the conditions of our model, the first order conditions for this problem are the same as those in (3). Then if a pair $(\overline{p}, \overline{x})$ is a price-allocation equilibrium, there exists a $\overline{\lambda}$ such that $\overline{x}(s) = \overline{x}(s, \overline{\lambda})$; solves (6), and $\overline{p}(s) = \gamma(s, \overline{\lambda})$ solves (4) for *a.e.s.*

Moreover we have the following proposition:

PROPOSITION 2. *A pair $(\overline{p}, \overline{x})$ is an equilibrium if and only if there exists $\overline{\lambda} \in \Delta^{n-1}$ such that $\overline{x}(s) = \overline{x}(s, \overline{\lambda})$; solves (6) and $\overline{p}(s) = \gamma(s, \overline{\lambda})$, solves (4) for a.e.s. and $e(\overline{\lambda}) = 0$.*

PROOF. Suppose that $\overline{x}(\cdot, \overline{\lambda})$ solves (6) and $\gamma(s, \overline{\lambda})$ solves (4). If for $\overline{\lambda} \in \Delta^{n-1}$ we have that $e(\overline{\lambda}) = 0$, then the pair $(\overline{p}, \overline{x})$, with $\overline{p} = \gamma(\cdot, \overline{\lambda})$ and $\overline{x} = x(\cdot, \overline{\lambda})$, is an equilibrium. ■

Reciprocally, if $(\overline{p}, \overline{x})$ is an equilibrium, then is straightforward from definition that $e(\lambda) = 0$. From the first welfare theorem, there exists $\overline{\lambda} \in \Delta^{n-1}$ such that $\overline{x}$ is a solution for (6). Since p is an equilibrium price, it is a support for $\overline{x}$, i.e. if for some x we have that $u_i(x) \geq u_i(\overline{x})$, $i = \{1, \ldots, n\}$, strictly for some i, then $\langle \overline{p}, x_i \rangle > \langle \overline{p}, w_i \rangle$ and from the first order conditions we have that: $\overline{p}(s) = \gamma(s)$. The proposition follows.

Let be $S^{n-1}_{++} = \{\lambda \in R^n : \|\lambda\|^2 = \sum_{i=1}^{n} \lambda_i^2 = 1\}$.

From remark 2, with $\alpha = \frac{1}{\|\lambda\|^2}$, we can consider e defined on S^{n-1}_{++}.

We will give now the definition of the equilibrium set.

DEFINITION 8. *We will say that* λ *is an equilibrium for the economy if* $\lambda \in E$, *where* $E = \{\lambda \in S^{n-1}_{++} : e(\lambda) = 0\}$. *The set E will be called, the equilibrium set of the economy.*

A pair formed by a utility function and an endowment will be called a characteristic.

We will endow the set of characteristics $\boldsymbol{C} = \boldsymbol{U} \times \boldsymbol{M}$ with the following topology: $(U_n, w_n) \to (U, w)$ if for each compact $K \in R^l_{++}$

$$\|(U_n, w_n) - (U, w)\|_K \to 0 \text{ with } n \to \infty,$$

where

$$\|(U, w)\|_K = \|(U)\|_K + \|w\| = ess\ \sup_{s \in \Omega} \max_K (|U| + |\partial U| + |\partial^2 U| + \|w(s)\|)$$

This is a metrizable space and the induced metric can be taken as:

$$\|(U, w)\| = \sum_{N=1}^{\infty} 2^{-N} \frac{\|(U, w)\|_{K_N}}{1 + \|(U, w)\|_{K_N}},$$

where $K_N = \{z \in R^l : \frac{1}{N} \leq z_i \leq N\}$.

An economy $\mathcal{E}$ is a list $(U_i, w_i) \in \boldsymbol{C}$, $i \in I$, where I is a set of traders. Let Γ be the set of economies with characteristics in $\boldsymbol{C}$ such that zero is a regular value of its excess utility function.

That is, for any λ such that $e(\lambda) = 0$ we have that rank of the Jacobian of $e(\lambda)$ is $n - 1$, i.e.: *rank* $J[e(\lambda)] = n - 1$.

From Mas-Colell (1991), we know that Γ is open and dense in the set of economies. From now on we will work with economies in Γ.

Let be $T_\lambda S^{n-1}_{++} = \{\bar{\lambda} \in R^n : \bar{\lambda}\lambda = 0\}$, and $\Pi_{\bar{\lambda}}$ the orthogonal projection from R^n onto $T_\lambda S^{n-1}_{++}$. Since whenever $e(\bar{\lambda}) = 0$, $J[e(\bar{\lambda})]$ maps into $T_{\bar{\lambda}} S^{n-1}_{++}$ (to verify it differentiate $\lambda e(\lambda) = 0$). Therefore if $\bar{\lambda}$ is a regular value, $J[e(\bar{\lambda})]$ maps onto $T_{\bar{\lambda}} S^{n-1}_{++}$. Its determinant is equal to the determinant of the following matrix, see (Mas-Colell, 1985, B. 5.2):

$$[\Pi_{\bar{\lambda}} J[e(\bar{\lambda})]] = \begin{bmatrix} J(e(\bar{\lambda})) & \bar{\lambda} \\ -\bar{\lambda}^{tr} & 0 \end{bmatrix}.$$

Since $\Pi_{\bar{\lambda}}\, J[e(\bar{\lambda})]$ is an isomorphism from $T_{\bar{\lambda}}S^{n-1}_{++}$ onto $T_{\bar{\lambda}}S^{n-1}_{++}$, its determinant is not zero.

We will put $\operatorname{sign} J(e(\lambda)) = (+1) - 1$ according to whether *det* $[\Pi_T\, J(e(\lambda))](> 0) < 0$.

We are now in condition of stating our main result:

THEOREM 1. *Consider an economy in* Γ *with an infinite dimensional consumption set, and separable utilities satisfying the conditions in section 1), then:*

1) The cardinality of E is finite and odd,

2) If sign $J(e(\lambda))$ *is constant in E, there exists an unique equilibrium, where* $J(e((\lambda))$ *denote the Jacobian of the excess utility function.*

4. Examples of Economies with Uniqueness

In this section we consider some examples with uniqueness of equilibria.

Let $[J(e(\lambda))]_{ij}$ be the term in the row i and column j of the Jacobian of the excess utility function.

$$[J(e(\lambda))]_{ij} = \frac{\partial e_i(\lambda)}{\partial \lambda_j}.$$

Then

$$[J(e(\lambda))]_{ij} = \int_{\Omega} \frac{\partial\{\partial U_i(s, x_i(s, \lambda))\, [x_i(s, \lambda) - w_i(s)]\}}{\partial \lambda_j}\, dv(s) \tag{7}$$

where $\partial U_i = \left(\frac{\partial U_i}{\partial x_1}, \ldots, \frac{\partial U_i}{\partial x_l}\right)$ and $x_i(s, \lambda) = (x_{i1}(s, \lambda), \ldots, x_{il}(s, \lambda))$.

We have that

$$\frac{\partial\{\partial U_i\}}{\partial \lambda_j} = \left[\frac{\partial x_i(\lambda)}{\partial \lambda_j}\right] \partial^2 U_i \text{ with } \left[\frac{\partial x_i(\lambda)}{\partial \lambda_j}\right] = \left(\frac{\partial x_{i1}}{\partial \lambda_j}, \ldots, \frac{\partial x_{il}}{\partial \lambda_j}\right)$$

and

$$\partial^2 U_i = \begin{bmatrix} \partial^2 U_i / \partial x_1^2 & \partial^2 U_i / \partial x_1 \partial x_2 \dots & \partial^2 U_i / \partial x_1 \partial x_l \\ \partial^2 U_i / \partial x_2 \partial x_1 & \partial^2 U_i / \partial x_2^2 \quad \dots & \partial^2 U_i / \partial x_2 \partial x_l \\ \vdots & \vdots & \vdots \\ \partial^2 U_i / \partial x_l \partial x_1 & \partial^2 U_i / \partial x_l \partial x_2 \dots & \partial^2 U_i / \partial x_l^2 \end{bmatrix}.$$

Then, we obtain

$$[J(e(\lambda))]_{ij} = \int_\Omega \frac{\partial x_i}{\partial \lambda_j} [\partial^2 U_i(x_i(s, \lambda))\, [x_i(s, \lambda) - w_i(s)]^{tr} + (\partial U_i(s, x_i(s, \lambda))^{tr}] dv(s) \tag{8}$$

4.1. *Economies with Gross Substitutes Property*

Following Dana (1993) we give the following definition.

DEFINITION 9. *The excess utility function displays the so called "Gross Substitute" property if:*

$$\frac{\partial e_i(\lambda)}{\partial \lambda_j} (> 0) < 0 \ \textit{if.}\ (i = j)\ i \neq j.$$

This property is only formally similar to the Gross Substitutes property displayed in some cases for the excess demand function, and has no straightforward economical sense. However, we have that the excess utility function of a consumer is raised if it is in his social weight, and that it is lowered if the social weight of some other consumer is raised.

PROPOSITION 3. *If the excess utility function has the Gross Substitute property, then sign $J(e(\lambda))$ is constant.*

PROOF. Let A be the $(n-1) \times (n-1)$ northwestern submatrix of $\{J(e(\lambda)) + [J(e(\lambda))]^{tr}\}$. From the Gross Substitute property we can prove that A has dominant diagonal positive (see McKenzie, 1960). Let be $v_n = (0, \dots, 1)$, as rank $J(e(\lambda))$ is $(n-1)$, then for all vector z such that $zv_n = 0$ we have that $zJ(e(\lambda))z > 0$.

Now take $v \neq 0$ with $\lambda v = 0$ and let $v_\alpha = v + \alpha\lambda$; if $\alpha = -\frac{vv_n}{\lambda v_n}$, then $v_\alpha v_n = 0$. If $e(\lambda) = 0$ we have that $v_\alpha J(e(\lambda))v_\alpha = vJ(e(\lambda)) > 0$. That is if $e(\lambda) = 0$ then $J(e(\lambda))$ as a map from T_λ to T_λ is positive definite, then its determinant is positive.

Now theorem 1 guarantees uniqueness of equilibria. ■

4.1.1. Economies with One Good in Each State

Economies with one good in each state of the world and utility functions with the next property

$$\frac{\partial^2 U}{\partial x^2}(x - w) + \frac{\partial U}{\partial x} \geq 0, \tag{*}$$

have Gross Substitute property. See Dana (1993).

For the following two examples the above condition is satisfied.

Example 1. Suppose an economy with individual's utility:

$$u_i(x) = \int_\Omega U_i(x(s))g_i(s)dv(s), \text{ with } g_i : \Omega \to R^+ \text{ and } i = \{1, \ldots, n\}.$$

If $U_i(x)$ satisfies (*), then we have uniqueness.

For instance: $u_i(x) = \int_\Omega x(s)_i^\alpha e^{-rs}dv(s)$ with $0 < \alpha < 1$ and $r > 0$.

Example 2. Let us now consider economies with the following utilities:

$$u_i(x) = \int_\Omega [a_i(s) + b_i(s)x(s)]^{\alpha_i} dv(s).$$

Where $a_i(s) > -w(s)\, b_i(s) > 0$ and $0 < \alpha_i < 1$. For these economies (*) is satisfied.

Example 3. If the economy has one good in each state of the world, i.e. $x : \Omega \to R$, and if each agent has risk aversion smaller than one, then the economy display Gross Substitute property. See Dana (1993).

The condition $\partial[x\partial U] \geq 0$ is equivalent to risk aversion smaller than one. So, risk aversion smaller than one implies (*), then uniqueness of equilibrium follows.

Example 4. Consider the following optimal control problem:

Given an initial endowment $x > 0$, an investor wishes to choose an admissible pair (π, c) of portfolio and consumption processes, so as to maximize

$$V_{\pi, c} = E\int_0^T e^{-\int_0^s \beta(u)du} U(C_s)ds$$

over every pair (π, c) admissible for x (Karatzas and Sreve, 1988).

For U_i as in example 3 the solution is unique.

4.2. *Separable Utility in Goods and States*

Consider economies with good-separable utility functions, i.e.

$$\frac{\partial^2 U_h}{\partial x^i \partial x^k} = 0 \text{ for all } h \in \{1, \ldots, n\} \text{ and } i, k \in \{1, \ldots, n\},\ i \neq k.$$

Proposition 4. *Let U be a utility function that is both additively separable and good separable. If*

$$\begin{aligned} &\partial^2 U_h(s, x_h(s, \lambda))\,[x_h(s, \lambda) - w_h(s)]^{tr} \\ &+ [\partial U_h(s, x_h(s, \lambda))]^{tr} >> 0\ (<< 0) \qquad (9) \\ &\text{for all } h \in \{1, \ldots, n\} \text{ and } s \in \Omega, \end{aligned}$$

then we have uniqueness.

Proof. From the first order conditions, (2), we have that

$$\lambda_1 \partial U_1(s, x_1(s, \lambda)) \equiv \ldots \equiv \lambda_n \partial U_n(s, x_n(s, \lambda)$$

that is

$$\lambda_1 \frac{\partial U_1}{\partial x^k} \equiv \ldots \equiv \lambda_n \frac{\partial U_n}{\partial x^k} \text{ for all } k \in \{1, \ldots, l\}$$

where

$$x_h = (x_h^1, \ldots, x_h^l)\ 1 \leq h \leq n$$

Taking derivatives with respect to $\lambda_j (j \in \{1, \ldots, n\},)$ and recalling that $\partial^2 U_h / \partial x^i \partial x^k = 0$, it follows that

$$\lambda_1 a_{1k} \frac{\partial x_1^k}{\partial \lambda_j} = \ldots = \lambda_j a_{jk} \frac{\partial x_j^k}{\partial \lambda_j} + b_{jk} = \ldots = \lambda_n a_{nk} \frac{\partial x_n^k}{\partial \lambda_j} \tag{10}$$

where

$$a_{hk} = \frac{\partial^2 U_h}{\partial x^{k2}} \text{ and } b_{hk} = \frac{\partial U_h}{\partial x^k}.$$

Let $w^k(s)$ be the total endowment of good k.

From $x_1^k(s, \lambda) + \ldots + x_n^k(s, \lambda) = w^k(s)$ we obtain that

$$\frac{\partial x_1^k}{\partial \lambda_j} + \ldots + \frac{\partial x_n^k}{\partial \lambda_j} = 0 \tag{11}$$

From (8) we obtain the following equation

$$\frac{\lambda_1 a_{1k}}{\lambda_h a_{hk}} \frac{\partial x_1^k}{\partial \lambda_j} = \frac{\partial x_h^k}{\partial \lambda_j}, \text{ for all } h \neq j. \tag{12}$$

Replacing (10) in (9) and without loss of generality supposing that $j \neq 1$ and $j \neq h \neq 1$ gives

$$\frac{\partial x_1^k}{\partial \lambda_j} + \frac{\partial x_j^k}{\partial \lambda_j} + \frac{\partial x_1^k}{\partial \lambda_j} \left\{ \sum_{h \neq 1\, h \neq j} \frac{1}{\lambda_h a_{hk}} \right\} \lambda_1 a_{1k} = 0,$$

or equivalently

$$\frac{\partial x_j^k}{\partial \lambda_j} + \frac{\partial x_1^k}{\partial \lambda_j} \left\{ \sum_{h \neq j} \frac{1}{\lambda_h a_{hk}} \right\} \lambda_1 a_{1k} = 0. \tag{13}$$

From (8)

$$-\lambda_1 a_{1k} \frac{\partial x_1^k}{\partial \lambda_j} + \lambda_j a_{jk} \frac{\partial x_j^k}{\partial \lambda_j} = -b_{jk} \tag{14}$$

Finally, replacing (12) in (11) we obtain

$$\frac{\partial x_1^k}{\partial \lambda_j} = \frac{b_{jk}}{\lambda_j a_{jk} \lambda_1 a_{1k} \sum_h \frac{1}{\lambda_h a_{hk}}} < 0$$

$$\frac{\partial x_i^k}{\partial \lambda_j} < 0, \text{ for all } k \text{ and } i \neq j. \tag{15}$$

From the fact that $x(\lambda)$ is homogeneous of degree zero we obtain that

$$\frac{\partial x_j^k}{\partial \lambda_j} > 0, \text{ for all } k \text{ and } j. \tag{16}$$

Then (13) and (14) are sufficient conditions to obtain (8) of proposition 2. ■

If the economy has one good in each state of the world, we have that:

REMARK 4: If $x(s, \cdot) : \Delta \to R$, the fact that

$$x_i(s, \lambda_1, \ldots, \lambda_{k-1}, \cdot, \lambda_{k+1}, \ldots, \lambda_n)$$

is increasing for λ_i and decreasing if λ_k, $k \neq i$ has economical sense because $x(\lambda)$ is a solution of the social choice problem.

That is if the social weight of $i-th$ agent is increased, then the consumption bundle of the agent must also increase.

The following example illustrates proposition 4.

EXAMPLE 5. Economies with utility functions such that

$$\partial\left[x\frac{\partial U_i(s,x)}{\partial x}\right] \geq 0 \tag{17}$$

have a unique equilibrium price.

In order to prove this assertion recall that:

1) $w_i(s)$ is positive for all i and $s \in \Omega$,

2) the Hessian is a diagonal matrix with positive entries.

For instance, economies with the following utility functions

$$U(x) = \int_\Omega [\sum_{j=0}^{N} \mathrm{p}^j u_j(x_j)] d\mu(s)$$

with $x = x_1, \ldots, x_n$, $u_j(x_j) = x_j^{\alpha_j}$ and $0 < \mathrm{p} < 1$, $0 < \alpha_j < 1$, have Gross Subtitute property.

4.3. *Economies with Two Goods and Two Agents*

Let E be an economy with two goods an two agents (u_i, w_i), $i = \{1, 2\}$.
From the first order condition we have that:

$$\lambda_1 \partial U_1(x_1) \equiv \lambda_2 \partial U_2(x_2)$$

where $x_i = (x_i^1, x_i^2)$. Taking derivatives with respect to λ_1 in the above identity we obtain:

$$b_1^i + \lambda_1 a_1^{i1} \frac{\partial x_1^1}{\partial \lambda_1} + \lambda_2 a_1^{i2} \frac{\partial x_1^2}{\partial \lambda_1} = \lambda_2 a_2^{i2} \frac{\partial x_2^1}{\partial \lambda_1} + \lambda_2 a_2^{i2} \frac{\partial x_2^2}{\partial \lambda_1} \quad i = 1, 2 \quad (18)$$

where

$$a_i^{kj} = \frac{\partial^2 U_i}{\partial x^k \, \partial x^j} \text{ and } b_1^i = \frac{\partial U_1}{\partial x^i}; \quad i, j, k = 1, 2.$$

Let $w^k(s)$ be the endowment of good k. Then

$$x_1^k(s, \lambda) + x_2^k(s, \lambda) = w^k(s) \quad (19)$$

Hence

$$\frac{\partial x_1^k}{\partial \lambda_j} = - \frac{\partial x_2^k}{\partial \lambda_j} \quad (20)$$

Replacing (18) in (16):

$$(\lambda_1 a_1^{11} + \lambda_2 a_2^{11}) \frac{\partial x_1^1}{\partial \lambda_1} + (\lambda_1 a_1^{12} + \lambda_2 a_2^{12}) \frac{\partial x_1^2}{\partial \lambda_1} = - b_1^1$$

$$(\lambda_1 a_1^{21} + \lambda_2 a_2^{21}) \frac{\partial x_1^1}{\partial \lambda_1} + (\lambda_2 a_1^{22} + \lambda_2 a_2^{22}) \frac{\partial x_1^2}{\partial \lambda_1} = - b_1^2$$

Then

$$\frac{\partial x_1^1}{\partial \lambda_1} = \frac{1}{\nabla} \begin{vmatrix} -b_1^1 & \lambda_1 a_1^{12} + \lambda_2 a_2^{12} \\ -b_1^2 & \lambda_1 a_1^{22} + \lambda_2 a_2^{22} \end{vmatrix} \quad (21)$$

and

$$\frac{\partial x_1^2}{\partial \lambda_1} = \frac{1}{\nabla} \begin{vmatrix} \lambda_1 a_1^{11} + \lambda_2 a_2^{11} & -b_1^1 \\ \lambda_1 a_1^{21} + \lambda_2 a_2^{21} & -b_1^2 \end{vmatrix} \quad (22)$$

where ∇ is the determinant of a 2×2 Hessian matrix of a convex combination of diffentiably strictly concave functions, then ∇ is non negative. We suppose that $\nabla > 0$.

Since $x(\lambda)$ is homogeneous of degree zero (see lemma 2, section 5) we obtain that

$$sgn \frac{\partial x_j^k}{\partial \lambda_j} = - sgn \frac{\partial x_j^k}{\partial \lambda_i} \quad k = 1, 2 \quad \text{and} \quad j \neq i. \tag{23}$$

From (18) we obtain that:

$$\frac{\partial x_j^k}{\partial \lambda_j} = - \frac{\partial x_i^k}{\partial \lambda_j} \quad k = 1, 2 \quad \text{and} \quad j \neq i \tag{24}$$

The next proposition follows.

PROPOSITION 5. *If the sign of* $\frac{\partial x_j^i}{\partial \lambda_j}$ *is the same for all* $i, j = (1, 2)$ *and if*

$$sgn\, [\partial^2 U_i \cdot (x_i - w_i) + \partial U_i] \textit{ is constant} \tag{25}$$

then uniqueness follows.

PROOF. In this conditions the Gross Substitute property follows.

A sufficient condition to obtain (24) is that:

$$[\partial^2 U_i \cdot w_i] \leq 0 \text{ and } \partial \left[x \frac{\partial U_i(s, x)}{\partial x} \right] \geq 0 \qquad \blacksquare$$

EXAMPLE 6. Economies with $a_k^{ij} > 0 \;\; i \neq j$ and $k = 1, 2$ satisfying (24), have uniqueness of equilibrium.

EXAMPLE 7. Suppose an economy with

$$u_1(X) = \int_\Omega (a(s)x(s) + b(s)y(s))^\alpha d\mu(s)$$

$$u_2(X) = \int_\Omega (x(s)^\beta + y(s)^\gamma) d\mu(s).$$

Where $X = (x, y)$ $0 < \{\alpha, \beta, \gamma\} < 1$ with a and b integral functions: $\Omega \to R_{++}$. The endowments are $w_i = \{w_{ix}, w_{iy}\}$.

We obtain that

$$\partial^2 U_1(x) + \partial U_1 = \alpha^2 (ax + by)^{\alpha - 1}\{a, b\} > 0$$

$$\partial^2 U_1 w = \alpha(\alpha - 1)\,(ax + by)^{\alpha - 2}\{aw_{1x} + bw_{1y}, bw_{1x} + aw_{1y}\} \leq 0$$

Then

$$[\partial^2 U_1(x - w_1) + \partial U_1 > 0$$

From (19) and (20) it follows that

$$\frac{\partial y_1}{\partial \lambda_1} = -\frac{b}{\nabla}\,\alpha\,(ax + by)^{\alpha - 1}\lambda_2 \gamma(\gamma - 1) y_2^{\gamma - 2} > 0$$

$$\frac{\partial x_1}{\partial \lambda_1} = -\frac{a}{\nabla}\,\alpha(ax_1 + by_1)^{\alpha - 1}\lambda_2 \beta(\beta - 1) x_1^{\beta - 2} > 0.$$

These conditions are satisfied for u_2 because it is a separable utility function. Uniqueness follows.

Example 8. The same result is obtained with

$$u_1(X) = \int_\Omega \log[a(s)x(s) + b(s)y(s)]d\mu(s)$$

and u_2 is a separable utility function.

Analogously to the Walrasian *tâtonement*, the differential equation system $\dot{\lambda} = e(\lambda)$, (since $\dot{\lambda} \in T_\lambda S^{n-1}_{++}$) defines a dynamical system on S^{n-1}_{++}. Unfortunately its economical sense is not as rich as in the *tâtonement* process. However, if there is only one good available in each state of the world, it prescribes that the excess utility function of a consumer be raised (resp. lowered) if his social weights does (see remark 3).

Remark 5. As a straightforward application of theorem 1) we observe that if for each $\bar{\lambda} \in E$ there exists a neighborhood $N_{\bar{\lambda}}$, such that for all $\lambda_0 \in N_{\bar{\lambda}}$ the solution $\lambda(t, \lambda_0)$ of the $\dot{\lambda} = e(\lambda)$, converges to $\bar{\lambda}$ as $t \to \infty$, then there exists an unique equilibrium. To see this, observe that a necessary condition for $\lambda(t, \lambda_0) \to \bar{\lambda}$ is that the sign of $Je(\bar{\lambda})$ is a constant.

In some others cases we can prove uniqueness of equilibrium as straightforward application of the first order conditions. For instance if $u_i = \int_\Omega x_1^{a_1}(s) \ldots x_m^{a_m}(s) dv(s)$ for all i with $\sum_{j=1}^{n} a_i = 1$ and $a_i > 0$, then $e(\bar{\lambda}) = 0$, if and only if $\bar{\lambda} = \{\frac{1}{m} \cdots \frac{1}{m}\}$.

5. Proofs

The main tool that will be used to prove theorem 1 is the Poincaré Hopf theorem.

Let us recall it.

Poincaré Hopf Theorem. Let N be a compact n-dimensional C^1 manifold with boundary and f a vector field on N. Suppose that:

i) f points outward at ∂N [this means that $f(x)g(x) > 0$ for all $x \in \partial N$, where g is the Gauss map] and,

ii) f has a finite number of zeros.

Then, the sum of the indices of f at the different zeros equals the Euler characteristic of N.

For the definition of index of f at x (zero of f) and the Euler characteristic of N, see Mas-Colell (1985).

We need, also, the following lemmas:

Lemma 1. *The excess utility function is* C^1.

Proof. The Lagrange multiplier $\gamma(s, \lambda)$ and the Pareto optimal allocation $x_i(s, \lambda)$ are C^1 with respect to λ. To prove this affirmation let us consider the following system of equations:

$$\begin{aligned} \lambda_i \partial U_i(s, x(s, \lambda)) &= \gamma(s, \lambda) \\ \sum_{i=1}^{n} x_i(s, \lambda) &= \sum_{i=1}^{n} w_i(s) \end{aligned} \tag{26}$$

From the implicit function theorem, taking derivatives in the above system, with respect to x and γ, we obtain a matrix with the following form:

$$M = \begin{bmatrix} A & B \\ B^t & 0 \end{bmatrix}.$$

Where A is a $(nl) \times (nl)$ matrix: and B is a $(nl) \times l$ matrix.

$$A = \begin{bmatrix} U^1_{11} & \dots & U^1_{l1} & 0 & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & 0 & \dots & \dots & 0 \\ U^1_{1l} & \dots & U^1_{ll} & 0 & \dots & \dots & 0 \\ 0 & \dots & 0 & \ddots & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & U^n_{11} & \dots & U^n_{l1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & U^n_{l1} & \dots & U^n_{1l} \end{bmatrix}$$

and

$$B = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 1 \\ 1 & 0 & \dots & 1 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

That is, B is a $l \times (nl)$ matrix. ■

Claim. There is no vector $z = (v, w) \neq 0$ with $v \in R^{nl}$ and $w \in R^l$ such that $Mz = 0$.

Proof. Let v such that $Mz = 0$, then

$$B^{tr}v = 0 \tag{27}$$

and

$$Av + Bw = 0 \tag{28}$$

Then, from (26) and (27) we have that

$$v^{tr}Av = 0 \tag{29}$$

If $v \in ker\ B^{tr}$ then

$$\begin{array}{l} v_1 + v_{l+1} + \ldots + v_{(n-1)l+1} = 0 \\ v_2 + v_{l+2} + \ldots + v_{(n-1)l+2} = 0 \\ \vdots \quad \vdots \quad \vdots \quad \vdots \\ v_l + v_{2l} + \ldots + v_{nl} = 0 \end{array}$$

Observe that

$$\partial \sum_{i=1}^{n} \lambda^i U^i =$$

$$\{\lambda^1 \frac{\partial U^1}{\partial x_1}, \lambda^1 \frac{\partial U^1}{\partial x_2}, \ldots, \lambda^1 \frac{\partial U^1}{\partial x_l}, \ldots, \lambda^n \frac{\partial U^n}{\partial x_1}, \lambda^n \frac{\partial U^n}{\partial x_2}, \ldots, \lambda^n \frac{\partial U^n}{\partial x_l}\} =$$

$$= \{\gamma_1, \gamma_2, \ldots, \gamma_l, \ldots, \gamma_1, \gamma_2, \ldots, \gamma_l, \ldots, \gamma_1, \gamma_2, \ldots, \gamma_l\}.$$

Then

$$\partial\{\sum_{i=1}^{n} \lambda^i U^i\} \cdot v =$$

$$\gamma_1(v_1 + v_{l+1} + \ldots + v_{(n-1)l+1}) + \ldots + \gamma_l(v_l + v_{2l} + \ldots + v_{nl}) = 0 \qquad (30)$$

From (28), (29) and the strictly differentiably convexity of $\sum_{i=1}^{n} \lambda^i U^i$ we deduce that $v = 0$.

Then, since B is an injective matrix, from (26) $w = 0$. We have that $z = 0$. Proving our claim.

From the claim and the fact that $U_i(s, \cdot)$ is in a compact set of Δ, the lemma follows.

Lemma 2. *The excess utility function has the following properties:*

1) $e(\lambda)$ is homogeneous of degree zero;
2) $\lambda e(\lambda) = 0$, for all $\lambda \in R^n_{++}$;
3) there exists $k \in R$ such that $e(\lambda) << k\mathbf{1}$.
4) $\|e(\lambda)\| \to \infty$ as $\lambda_j \to 0$ for any $j \in \{1, \ldots, n\}$ and $\lambda \in \Delta^{n-1}$;
5) $J[e(\lambda)] : T_\lambda S^{n-1}_{++} \to T_\lambda S^{n-1}_{++}$

Proof. From remark 1, property 1) follows. Properties 2) and 4) follow from remark 2) and definition 7). Property 5) was proved immediately before definition 10).

To prove property 3, note that from equation (2) we can write

$$e_i(\lambda) = \int_\Omega \partial U_i(s, x_i(\lambda))\,[x_i(s, \lambda) - w_i(s)]dv(s).$$

From the concavity of U_i it follows that:

$$U_i(s, x(s, \lambda)) - U_i(s, w(s)) \geq \partial U_i(s, x(s, \lambda))\,(x_i(s, \lambda) - w(s)).$$

Therefore,

$$e_i(\lambda) \leq \int_\Omega U_i(s, x_i(s, \lambda)) - U_i(w_i(s))dv(s) \leq \int_\Omega U_i(\sum_{j=1}^{n} w_j(s))dv(s), \text{ for all } \lambda.$$

If we let

$$k_i = \int_\Omega U_i(\sum_{i=1}^{n} w_i(s))dv(s) \text{ and } k = \sup_{1 \leq i \leq n} k_i,$$

property 3 follows. ■

We can now prove the following lemma:

LEMMA 3. *The excess utility function is an outward pointing vector field at the boundary of* S_{++}^{n-1}.

PROOF. From property 2 of lemma 2 it follows that $e(\lambda) \in T_\lambda S_{++}^{n-1}$.

To prove that $e(\lambda)$ is an outward pointing vector field, let us now define z_i

$$z_i = \lim_{\lambda^m \to \lambda \in \partial S_{++}^{n-1}} \frac{e_i(\lambda^m)}{\|e(\lambda^m)\|}.$$

By property 3 we know that there exists $k \in R$ such that $e_i(\lambda) \leq k$ and by Property 4, $\|e(\lambda)\| \to \infty$. Then we conclude that $z_i \leq 0$. ■

Furthermore, z_i could be different from zero only if λ_i were zero. This follows from the fact that if λ_i is different from zero, then we can write

$$e_i(\lambda^m) = \frac{-1}{\lambda_i^m} \sum_{j \neq i} e_j(\lambda^m) \geq -\frac{kn}{\lambda_i^m}.$$

Letting $k' = -kn / \lambda_i^m$, we have that $k' \leq e_i(\lambda^m) \leq k$. Hence $z_i = 0$.

Strictly speaking, we have proved that we have a continuous outward pointing vector field for *almost any point* in the boundry of S_{++}^{n-1}. The excess utility function has similar properties to those of the excess demand function. Mas-Colell (1985) proves that for excess demand functions there is an homotopic in ward vector field for *all points* of the boundary S_{++}^{n-1}. In our case, with an analogous proof, we can obtain an homotopic outward vector field for excess utility functions.

PROOF OF THEOREM 1. Since S_{++}^{n-1} is homeomorphic to the $(n-1)$-dimensional disk, its Euler characteristic is one.

The equilibrium set E is a compact set. Moreover, from the fact that zero is a regular value of e, we have that E is a finite set. On the other hand, $e(\lambda)$ is a C^1 vector field on the tangent space pointing outward at the boundary of S_{++}^{n-1}. Then we can apply the Poincaré Hopf theorem.

In our case, the index of the vector field e at $\lambda \in E$ is the sign of determinant of $J[e(\lambda)]$.

So, we obtain that:

$$1 = \sum_{\{\lambda : e(\lambda) = 0\}} sign\ det\ J(e(\lambda)).$$

The theorem follows by simple cardinality arguments. ■

6. Mitjushim-Polterovich Condition

In this section we shall show that if there exists a demand function, the Mitjushim-Polterovich condition (MP) (Mitjushim-Polterovich, 1978) can be applied for economies with a Banach space as consumption space. Since strict monotonicity of individual demand functions with respect to a fixed vector implies that the weak axiom of revealed preference holds for the aggregate, then MP condition guarantees convexity of the equilibrium set also in models involving infinite dimensional linear spaces. Furthermore, if the economy is distributive then the MP condition guarantees uniqueness of equilibrium.

We recall that the MP condition says that: *if the C^1 demand function $h : R_{++}^l \to R_{++}^l$ is generated by the C^2, monotone, concave utility func-*

tion $u : R^l \to R$, *then a sufficient condition for the strict monotonicity of* h *is that*

$$\sigma(x) = -\frac{\langle x, \partial^2 u(x)x\rangle}{\langle x, \partial u(x)\rangle} < 4, \text{ for all } x.$$

6.1. *Some Considerations About the Demand Function*

We say that the demand function $\Phi(p, w)$ is defined in (p, w) if it exists as a solution of $\sup_{\{x \in A,\, px = w\}} u(x)$. Where A is a closed and convex subset of the commodity space B.

REMARK 6. In economies with infinite dimensional consumption spaces, demand functions with strong properties are rare. Moreover, since the agent's budget may not be compact we can not guarantee the existence of demand functions. See Araujo (1987).

6.2. *Monotone Operators*

In order to obtain the main result of this section, we will consider some properties of monotone operators in Banach spaces.

Let B be a real Banach space, let B^* be the dual space of B, and let $F : B^* \to B$ be an operator.

DEFINITION 10. *F is called monotone (strictly) if*

$$\langle F(p) - F(q), p - q\rangle \leq 0 (< 0) \text{ for all } p, q\ (p \neq q) \in B^*.$$

DEFINITION 11. *If the operator $F : B^* \to B$ defined on an open set $D(F)$ has a Gateaux differential at p, F is said to be locally strictly monotone at the point $p \in D(F)$ if the Gateaux-derivative $F'(p) : B^* \to B^{**}$, where B^{**} is the second dual space for B, is negative definite. That is*

$$\langle F'(p)h, h\rangle < 0 \quad \textit{for all } h \in B^* \quad \textit{with } h \neq 0.$$

LEMMA 4. *Let F be a Gateaux differentiable operator in a convex set $D \subseteq B^*$; then the negative definiteness of $F'(p)$ implies the strict monotonicity of F.*

PROOF. For p and $q \in D$, let us consider $v = p - q$ and $p(\alpha) = \alpha p + (1 - \alpha)q$, with $0 < \alpha < 1$.

Define the function $g : [0, 1] \to R$ by $g(\alpha) = \langle v, (F(p(\alpha)) - F(q))\rangle$. We obtain: $g'(\alpha) = \langle v, F'(p(\alpha))v\rangle$. Since $g(0) = 0$, and $g'(\alpha) < 0$, then $g(1) = \langle (p - q), (F(p) - F(q))\rangle < 0$. ■

DEFINITION 12. *We say that F is monotone with respect to the normalizing vector* $e \geq 0$ *if* $(F(p) - F(q))\,(p - q) \leq 0$ *whenever* $\langle p, e\rangle = \langle q, e\rangle = 1$ *(it is strictly monotone if the inequality is strict for all* $p \neq q$*).*

6.3. *The Model*

We shall consider a pure exchange economy with uncertainty in the states of the world Ω, and we shall treat uncertainty as a probability space (Ω, S, v), where S is the σ-algebra of subsets of Ω that are events, and v a probability measure. In each state of the world, there are n commodities available for consumption. There are m agents, each characterized by his consumption space $X = \prod_{j=1}^{n} X_j$, where $X_j \subseteq B^+$, closed and convex and B^+ is the positive cone of a Banach space, i.e. $B^+ = \{x \in B : x \geq 0\}$; his utility function $u_i : X \to R$ and his endowments $e_i \in X$, $i = \{1, \dots, m\}$.

Utility functions u_i fulfills the following properties:

H1) u_i is increasing and has continuous Gateaux derivatives up to second order.

H2) The second Gateaux derivative is a negative definite bilinear form.

H3) u_i satisfies the Inada condition, that is, its Gateaux derivative is inifinite at zero, for each direction $h \in X$, $\|u'(x)\| \to \infty$, as $x_j \to 0$ for any $j \in \{1, \dots, n\}$ and $x \in X$.

Every consumer i has strictly positive endowments $e_i = \theta_i e$, where e is a fixed strictly positive vector in X and 0_i is a positive number. The normalized set of prices is a fixed, bounded, convex, relatively open set $P = \{p : p \in X^* = \prod_{j=1}^{n} X_j^*, X_j^* \subset B^*; \langle p, e\rangle = 1\}$, our set of unnormalized prices is $\bar{P} = \{\lambda p\,; p \in P : \delta < \lambda < \frac{1}{\delta}\}$, where δ is a fixed number. The budget set for each consumer is the set $X_i = \{x \in X : \langle p, x\rangle \leq 1\}$. We will denote for $F : P \to X$ the demand function for each consumer when the endowment is fixed.

A comment is in order here, since the positive cone of B^* has empty interior, thus we are allowing for some non-positive prices. Since the view taken in this chapter is that the demand function exists, moreover there exists some p in B^*, such that p is a zero for the excess of demand function, then the uniqueness of equilibrium is not conceptually related to non positive prices.

6.4. MP *Condition and Demand Function*

The MP condition is posed at individual level, the question is: is the MP condition a sufficient condition on the preferences of consumer to guarantee that individual excess demand is monotone with respect to his endowment, for economies with a Banach space as commodity space? We give an affirmative answer.

Note that the property of monotone excess demand is a strong property to focus uniqueness of equilibrium, for instance the Weak Axiom of Revealed Preference is a more general condition that on regular economies, that is economies with isolated equilibria, by itself guarantees uniqueness of equilibria. However the first property tourns out more convenient to obtain uniqueness of equilibrium, because aggregates better across consumers.

Let X be a convex and closed subset of a Banach space B, and let $F : P \to X$ be a demand function. Suppose that the income is fixed to one, that is $\langle p, F(p)\rangle = 1$, for all $p \in P$.

We will use the following notation:

$\langle h, \partial u(x)\rangle$, for the Gateaux derivative at x with increment h.

$\langle h, \partial^2 u(x)k\rangle$, for the second Gateaux derivatives at x with increments h and k.

THEOREM 2. *If a demand function $F : P \to X$ is generated from a utility function $u : X \to R$ that satisfies H1, H2, H3, then a sufficient condition for the strict monotonicity of F in each $p \in P$ is*

$$\sigma(x) = -\frac{\langle x, \partial^2 u(x)x\rangle}{\langle x, \partial u(x)\rangle} < 4, \text{ for all } x \in X : \langle x, \partial u\rangle \neq 0. \quad (31)$$

PROOF. Define for $\langle x, \partial u(x)\rangle \neq 0$, $g : B \to B^*$, giving for the following identity:

$$g(x) = \frac{1}{\langle x, \partial u(x) \rangle} \partial u(x). \tag{32}$$

Let F be the demand function, i.e. $F(p) \in argmax\{u(x)\ s.t.\ \langle p, x \rangle = 1, x \in X\}$.

From *H3* we know that $F(p)$ is positive in each component for almost every $s \in \Omega$, then the following first order condition holds, $\partial u(F(p)) = \gamma p$. See Araujo, Monteiro (1989).

Claim. For all $p \in P$ we have that $g(F(p)) = p$.

Proof: For all $\alpha \in X$ we have

$$\langle \alpha, g(F(p)) \rangle = \frac{1}{\langle F(p), \partial u(F(p)) \rangle} \langle \alpha, \partial u(F(p)) \rangle.$$

Since $\langle p, F(p) \rangle = 1$, the above equality is well defined. From the first order condition we obtain $\langle F(p), \partial u(F(p)) \rangle = \gamma$. We also know that $\langle \alpha, \partial u(F(p)) \rangle = \gamma \langle \alpha, p \rangle$. Finally the following identity holds, $\langle \alpha, g(F(p)) \rangle = \langle \alpha, p \rangle$ for all $\alpha \in X$. Hence $g(F(p)) = p$, and then the claim is proved.

From claim we have that for all $p \in P$ g is the inverse of F.

Recall that if A is a strictly monotone operator, then the inverse operator is strictly monotone. See Zeidler (1990).

We will prove that g is a monotone operator in $F(p)$. To see this recall that, $g : X \to L(X, R)$, then $\partial g : X \to L(X, L(X, R))$, then $\partial g(x)$, $X \times X \to R$ is a bilinear form, for all $x \in P$ and $\langle x, \partial u(x) \rangle \neq 0$. See Kolmogorov, Fomin (1972). If we prove that $\langle v, \partial g(x) v \rangle < 0$, then g will be a strictly monotone function restricted to $F(p)$, and then F will be strictly monotone.

Now following Mas-Colell (1988), we will show that g is a strictly monotone function. Since $\partial u(x) = \gamma p$ and $\langle p, x \rangle = 1$, we obtain that $\gamma = \langle x, \partial u(x) \rangle$. Denoting $q = \partial u(x)$ and $A = \partial^2 u(x)$, differentiating g, we obtain that:

$$\partial_j g_i(x) = \frac{1}{\langle x, q \rangle} \partial^2_{ij} u(x) - \frac{1}{\langle x, q \rangle^2} \left[\partial_i u(x) \partial_j u(x) + \partial_i u(x) \sum_h x_h \partial_{hj} u(x) \right]$$

$$\partial g(x) = \frac{1}{\langle x, q \rangle} A - \frac{1}{\langle x, q \rangle^2} \left[q q^{tr} + q (Ax)^{tr} \right]$$

1) If $\langle v, q\rangle = 0$ then $v\,\partial^2 g(x)v = \frac{1}{\gamma}\langle v, Av\rangle < 0$.
2) If $\langle v, q\rangle \neq 0$, it suffices to consider $\langle v, q\rangle = \gamma$, then

$$\langle v, \partial g(x)v\rangle = \frac{1}{\gamma}\langle v, A(v-x)\rangle - 1.$$

Since $\langle v, A(v-x)\rangle = \langle (v-\frac{1}{2}x), A(v-\frac{1}{2}x)\rangle - \frac{1}{4}\langle x, Ax\rangle \leq -\frac{1}{4}\langle x, Ax\rangle$, the inequality follows from the concavity of the utility function u. Hence $\langle v, \partial g(x)v\rangle \leq -\frac{1}{4}\frac{\langle x\partial^2 u(x)x\rangle}{\langle x, \partial u(x)\rangle} - 1 < 0$. Then if $\sigma(x) < 4$, $\partial g(x)$ will be negative definite.

The theorem follows. ■

6.5. *Uniqueness Using* MP

If each individual demand function F_i is a strictly monotone function with respect to the endowment e_i and if E_i is the excess demand function, since $\langle p-q, E_i(p) - E_i(q)\rangle = \langle p-q, F(p)_i - F_i(q)\rangle$ then E is a strictly monotone function respect to the vector e_i.

Definition 13. *An economy is said to be distributive if the initial endowments are collinear ($e_i = k_i e$ where k_i is a constant for each $i = \{1, \dots, n\}$ and $e \in R^n$ is a fixed vector) or the distribution of income is price-independent.*

Since monotonicity of the individual demand function with respect to the same vector is preserved in aggregate, then the following theorem is straightforward.

Theorem 3. *Let $\mathcal{E}$ be a distributive economy, which satisfies the* MP *condition, then $\mathcal{E}$ has uniqueness of equilibrium.*

Proof. From theorem 3 we know that the MP condition guarantees the strict monotonicity of the individual demand function with respect to the normalizing vector e, then the excess utility function is strictly monotone with respect to the same vector, and since the economy is distributive we obtain a monotone aggregate excess demand function with respect to the vector e. Hence uniqueness of equilibrium follows. ■

REMARK 7. In distributive economies, if the individual demand function are monotones weak axiom holds.

To see this suppose that $E: P \to X$ is the excess demand function, with $\langle q, E(p)\rangle \leq 0$, and that $p \neq q$, since $\langle p, e\rangle = \langle q, e\rangle = 1$, from $\langle p-q, E(p)-E(q)\rangle < 0$, $\langle p, E(q)\rangle > 0$, follows.

REMARK 8. As long as the distribution of income is price independent, the MP condition guarantees the uniqueness of equilibrium. Nevertheless, the situation changes drastically when initial endowments are not collinear. For instance consider the following example.

EXAMPLE. Consider the following two consumer, two goods in each state of the world economy,

$$u_1(x) = \int_\Omega [x_{11}(s)]^{\frac{1}{2}} + \varepsilon[x_{12}(s)]^{\frac{1}{2}} d\lambda(s),$$

$$u_2(x) = \int_\Omega \varepsilon[x_{21}(s)]^{\frac{1}{2}} + [x(s)_{22}]^{\frac{1}{2}} d\lambda(s).$$

Where λ is the Lebesgue measure, and the set of the world states are a probability space $\{[01], \boldsymbol{B}\}$, with $\boldsymbol{B}$ the Borel σ-algebra.

Endowments are $w_1(s) = (w_{11}(s), 0)$; $w_2(s) = (0, w_{22}(s))$. Suppose that the consumption space is a closed and bounded subset of a positive cone in $L^p(\lambda) 1 \leq p < \infty$, then we obtain a weakly compact budget set. Since utility functions are weakly upper semicontinuous, the existence of demand function follows. From the Inada condition we obtain a positive for almost every state of the world demand function, then the first order condition holds.

The aggregate demand function is:

$$F(p) = \left(\frac{p_2 w_{11}}{p_2 + p_1 \varepsilon^2} + \frac{\varepsilon^2 p_2^2 w_{22}}{p_1^2 + p_1 p_2 \varepsilon^2}, \frac{\varepsilon^2 p_1^2 w_{11}}{p_2^2 + p_1 p_2 \varepsilon^2} + \frac{p_1 w_{22}}{p_1 + p_2 \varepsilon^2} \right),$$

for each state of the world.

The MP coefficients are $\sigma_1(x) = \sigma_2(x) = \frac{1}{2}$, nevertheless the excess demand function is not a monotone function. To see this, consider $\langle p-q, E(p)-E(q)\rangle$, where $E(p)$ is the excess demand function,

$p = \alpha(1, \frac{1}{2})$, $q = \beta(1, 1)$ where α and $\beta \in L^q(\lambda)$, $\frac{1}{p} + \frac{1}{q} = 1$, $6\alpha(s) > \beta(s)$ *a.e.s.* and endowments $w_{22} = \frac{1}{2} w_{11}$. For $\varepsilon = \frac{1}{\sqrt{2}}$, we obtain

$$\langle p - q, E(p) - E(q) \rangle > 0.$$

6.6. *Examples with Uniqueness of Equilibrium*

Example 1. Regular and distributive economies with utility functions $u : L_p^+ \to R$, $1 \le p \le \infty$, satisfying the following condition

$$-\frac{x(s) \cdot \partial^2 U(s, x(s)) x(s)}{x(s) \cdot \partial u(s, x(s))} < 4,$$

a.e.s. $\in \Omega$ and for all $x \in P$, and $\langle x, \partial u(x) \rangle \neq 0$ have equilibrium uniqueness because

$$\int_\Omega x(s) \cdot \partial^2 U(s, x(s)) x(s) + 4x(s) \cdot \partial U(s, x(s)) dv(s) > 0.$$

Then if follows that

$$-\frac{\langle x, \partial^2 u(x) \cdot x \rangle}{\langle x, \partial u(x) \rangle} < 4$$

and therefore the MP conditions is satisfied.

Example 2. Let X be a $L_p^+(v)$, $1 \le p \le \infty$ space. All distributive and regular economies with state separable and good separable utilities, i.e. $U(x) = \sum_{j=1}^{l} U_j(s, x(s))$ (see subsection 2.3.1), that also satisfy

$$-\frac{x_j(s) \partial^2 U_j(s, x_j(s))}{\partial U_j(s, x_j(s))} < 4 \text{ for all } j \in (1, \dots, l), \ a.e.s. \in \Omega$$

and $x \in P$, $\langle x, \partial u(x) \rangle \neq 0$ have equilibrium uniqueness.

If the above inequality is true we have that

$$\sum_{j=1}^{l} x_j(s) \partial^2 U_j(s, x_j(s)) x_j(s) + 4x_j(s) \partial \, U_j(s, x_j(s)) > 0 \quad a.e.s. \in \Omega$$

or equivalently

$$x(s) \partial^2 U(s, x(s)) x(s) + 4x(s) \partial U(s, x(s)) > 0.$$

Then
$$\int_{\Omega} x(s)\partial^2 U(s, x(s))x(s) + 4x(s)\partial U(s, x(s))dv(s) > 0$$

That is $\langle x, \partial^2 u(x)x\rangle + \langle 4x, \partial u(x) > 0\rangle$, which is equivalent to the MP condition.

It was shown in the last two examples that if for $a.e.s. \in \Omega$ the coefficient of relative risk aversion is less than 4, then it follows that the cardinality of equilibria is one.

References

Araujo, A. (1987). "The Non-Existence of Smooth Demand in General Banach Spaces", *Journal of Mathematical Economics*, vol. 17, pp. 1-11.

———, and P. K. Monteiro (1989). "Equilibrium without Uniform Conditions", *Journal of Economic Theory*, vol. 48, pp. 416-427.

Becker, G. S. (1971). *Economic Theory*, Alfred A. Knopf, New York.

Dana, R. A. (1993). "Existence, Uniqueness and Determinacy of Arrow-Debreu Equilibria in Finance Models".

Karatzas, I., and S. E.Sreve (1988). *Brownian Motion and Stochastic Calculus*, Springer Verlag.

Kehoe, T. J. (1991). "Computation and Multiplicity of Equilibria", in W. Hildenbrand and H. Sonnenschein (eds.), *Handbook of Mathematical Economics*, vol. IV, New York, North Holland..

Kolmogorov, A. N., and S. V. Fomin (1972). *Elementos de la teoría de funciones y del análisis funcional*, Moscú, Mir.

McKenzie, L. (1960). "Matrices with Dominant Diagonal on Economic Theory", in K. Arrow, S. Karlin, and P. Suppes (eds.), *Mathematical Methods in the Social Sciences*, pp. 47-62.

Mas-Colell, A. (1985). *The Theory of General Equilibrium, a Differentiable Approach*, Cambridge University Press.

——— (1986). "The Price Equilibrium Existence Problem in Topological Vector Latices", *Econometrica*, vol. 54, pp. 1039-1053.

——— (1988). "On the Uniqueness Equilibrium Once Again", (preprint).

——— (1991). "Indeterminacy in Incomplete Market Economies", *Economic Theory*, vol. 1, pp. 45-62.

Mitjushim, L. G., and V. W. Polterovich (1978). "Criteria for Monotonicity of Demand Functions", *Ekonomika i Matematicheskie Methody*, vol. 14, pp. 122-128.

Zeidler, E. (1990). *Nonlinear Fmctional Analysis and its Applications*, Springer-Verlag.