

**JOINT SMOOTHING OF GDP AND UNEMPLOYMENT
WITH A BIVARIATE HP FILTER**

**SUAVIZAMIENTO CONJUNTO DEL PIB Y LA TASA
DE DESEMPLEO CON UN FILTRO HP BIVARIADO**

Alejandro Islas

Víctor M. Guerrero

Instituto Tecnológico Autónomo de México

Resumen: Se considera el problema de estimar conjuntamente las tendencias y los ciclos no-observables de una serie de tiempo bivariada, en el contexto macroeconómico de la Ley de Okun, para relacionar las brechas del producto y la tasa de desempleo. La estimación se realiza con un filtro de Hodrick-Prescott bivariado que estima simultáneamente la correlación entre los ciclos. La contribución principal de este enfoque es que se puede controlar la suavidad de las tendencias al fijar un parámetro de suavizamiento en forma apropiada. La ilustración empírica utiliza datos del PIB y de la tasa de desempleo de Estados Unidos.

Abstract: The problem of jointly estimating unobserved trends and cycles of a bivariate time series considered here appears within the macroeconomic context of Okuns Law when relating output gaps and the unemployment rate. Joint estimation of the output and unemployment trends is carried out by applying a bivariate time series filter that allows for simultaneous estimation of the correlation between cycles. The estimation procedure employed is based on an extension of the univariate Hodrick-Prescott filter and its application in practice is relatively easy. The main contribution of our approach is that we can control the amount of smoothness in the trends by fixing the smoothing parameter. The empirical application uses data on U.S. real gross domestic product and the unemployment rate.

Clasificación JEL/JEL Classification: C32, E23, E24, E32

Palabras clave/keywords: business cycle, correlations, cyclical unemployment, signal extraction, smoothness index, ciclo de negocios, correlación, desempleo cíclico, extracción de señal, índice de suavidad

Fecha de recepción: 13 VI 2017

Fecha de aceptación: 13 XI 2017

Estudios Económicos, vol. 34 núm. 1, enero-junio 2019, páginas 3-24

1. Introduction

This paper deals with the problem of jointly smoothing output and unemployment rates. The problem can be approached from several standpoints, *e.g.*, by considering a single or a multiple time-series setting. Here, we concentrate on a bivariate time series approach that assumes that the observed time series vector is composed of an unobserved vector of trends plus a vector of cycles, and that takes into account the correlation between output and unemployment cycles. In the macroeconomic literature of real business cycle theory employed here, the cycle of output is usually called the output gap, since it corresponds to the difference between potential output, *i.e.*, the underlying trend of output, and actual output. Laxton and Tetlow (1992) provided a historical overview of estimation procedures of potential output and found that basically two approaches had been employed since the 1980s: (1) structural approaches that rely on a structural economic model, as in Ford and Rose (1989) and Adams and Coe (1990); and (2) stochastic approaches such as that underlying the Hodrick and Prescott (HP, 1997) filter.

Laxton and Tetlow (1992) combined those two approaches and proposed a semi-structural technique which is called the Hodrick-Prescott Multivariate filter (HPMV). Boone (2000) and Chagny and Lemoine (2002) apply this approach in their works. However, the HPMV is not a true multivariate filter, but a multiple time series filter. This difference is akin to that of a multiple regression where there is only one dependent variable to be explained by several independent ones, while multivariate regression considers several dependent variables to be explained simultaneously by one or more independent variables.

We propose a different semi-structural technique that is based on a true multivariate time series filtering method that we call a Bivariate Hodrick-Prescott filter (BHP), in order to decompose the time series vector into trend plus cycle, taking into account the correlation between cycles of the two series. Our basic aim is to estimate the cost of unemployment in terms of potential output according to Okun's Law. An approach similar to ours is that of Dermoune, Djehiche and Rahmania (2009) except for the fact that we emphasize the smoothness of the trend, which is the most important feature of a time series trend. We can measure the smoothness and control it by fixing the value of a smoothing parameter, in order to obtain comparable results for the different series considered by the vector of variables under study.

In his seminal paper, Okun (1962) estimated that a 1-percentage-point increase in unemployment would induce a decline in output growth of about 3.3 percent. We want to stress that the underlying assumption for measuring the impact of unemployment on potential output is that the unemployment rate summarizes –or is correlated with– other variables such as: average hours worked, participation rates and labor productivity. In other words, unemployment “...can be viewed as a proxy variable for all the ways in which output is affected by idle resources...” (Okun, 1962: 2). This assumption is very important for obtaining and predicting a fixed coefficient between unemployment change and output growth; otherwise, one should not expect this coefficient to be fixed. Furthermore, it implies that technological advances, changes in labor market institutions, variations in participation rates and demographic shifts, among other things, would induce changes in the coefficient.

The literature on this topic identifies three different techniques for estimating Okun’s coefficient: (i) by means of a simple two-step procedure; (ii) as part of a bivariate trend-cycle model for output and unemployment; and (iii) by assuming that the coefficient varies over time. The conventional method for estimating Okun’s coefficient involves a two-step procedure: First the permanent component of the output and unemployment series is estimated and removed and then the correlation between the transitory components is estimated. The permanent component of the series can be obtained using techniques that range from estimating the trend component by Ordinary Least Squares (OLS) to using the HP filter. In some cases, the permanent component is simply eliminated by taking first differences of the series. When the series is stationary this first step is omitted. Once the permanent component has been estimated, the transitory component is obtained by subtracting the estimated permanent component from the observed series. In the second step Okun’s coefficient is estimated by OLS.

Sinclair (2009) claims that the above methodology provides a biased and inefficient coefficient estimate for two reasons. First, since the permanent and transitory components of the two series are correlated, it is more efficient to estimate their cyclical components jointly. Second, to the extent that the measurement error of the independent variable is correlated with the measurement error of the dependent variable, OLS estimates are biased and inconsistent. Thus, a better approach uses the estimate of the correlation rather than the correlation of the estimates.

A second technique, proposed by Clark (1987), takes this criti-

cism into account and involves the use of a bivariate model to jointly estimate the permanent and transitory elements of unemployment and output. Such an approach was proposed as a reaction to Nelson and Plosser's (1982) methodology to remove non-stationarity by first differencing, assuming that the trend behaves as a random walk with drift rather than as a deterministic straight line. In fact, Clark (1987) pointed out that two shortcomings of this approach are, first, tests for non-stationarity in trend have very little power against plausible alternatives; second, the analysis is based on the strong assumption that the auto-covariance function for the first difference of output is exactly zero after lag one. Thus, he proposed a new method for analyzing United States output and unemployment data by decomposing each series into a non-stationary trend and a stationary cyclical component. The framework for his analysis is a state-space model that allows for a fairly general specification of the trend component.

A few years later, Clark (1989) used Kalman filtering and Maximum Likelihood to estimate the non-stationary permanent and stationary cyclical components of output growth and unemployment for six developed economies (Canada, France, Germany, Japan, United Kingdom and United States), and found strong evidence that the estimated output's stationary component is closely related to unemployment's cyclical component. Evans (1989) used a bivariate Vector Autoregressive (VAR) model to describe the output-unemployment dynamics, to estimate the degree of persistence in output innovations, and to decompose output into trend and cycle. He concluded that the bivariate analysis indicated the existence of feedback, as well as a negative contemporaneous correlation between output growth and unemployment innovations.

The third estimation technique was developed because there is some empirical evidence that the coefficient has not remained constant over time. In fact, since the mid-1990s, an increasing number of studies have investigated whether Okun's coefficient is stable over time. For example, Prachowny (1993) argued that the 3:1 ratio of output to unemployment holds only because other factors, including weekly hours worked, induced a tendency for labor supply and productivity to rise as well. An important conclusion of Prachowny's paper is that if any of the other factors change then, other things being equal, the coefficient linking output to unemployment will change as well.

In this paper we do not seek to compare the previous approaches, which are mentioned just for completeness but, as mentioned before, we aim to estimate the unobserved trend and cycle components of

output and unemployment, as well as Okun's coefficient, simultaneously. To do that, we propose the use of a BHP filter that arises as a simple extension of the original Penalized Least Squares (PLS) problem that gave rise to the HP filter. As such, the BHP filter provides a straightforward solution to the estimation problem, without requiring a full statistical model specification with its corresponding assumptions, but just the existence of second order moments.

Thus, our proposal is in line with the aforementioned second type of techniques, but we employ a new way of calibrating the smoothing parameter involved, that basically consists of fixing a desired percentage of smoothness to be achieved by both the output and unemployment trend estimates. This is the approach followed by Guerrero (2008) in a univariate time series setting; by doing so we obtain comparable trends that are estimated jointly with the correlation coefficient between cycles. The results obtained in the empirical application to United States data are seen to be sensible and resemble those produced by a complete structural model specification, with the advantage that our results are obtained with relative ease and do not require validating model assumptions, since no statistical model specification is employed, but only a filtering technique. Thus, our approach emphasizes the idea of simplicity over a complete model specification.

The organization of the rest of this article is as follows. Section 2 shows the derivation of the BHP filter and its solution by means of a state-space representation. In section 3 we illustrate the use of the proposed BHP filter with a real application to United States data on real GDP and unemployment. A comparison of our results is made with those obtained previously by Sinclair (2009). In section 4 we conclude with some final remarks.

2. Statistical Methodology

Let us define a bivariate column vector $Z_t = (Z_{1t}, Z_{2t})'$ where prime denotes the transpose of that vector throughout this paper. We assume that the bivariate time series under study is given by the signal-plus-noise model

$$Z_t = \tau_t + \eta_t \text{ for } t = 1, \dots, N, \quad (1)$$

Where Z_t is the observed vector, τ_t is also a bivariate vector denoting its trend (signal) and η_t is the bivariate vector denoting

its cycle (noise). Of course, as in the univariate case, such a representation does not correspond with the data generation process, but it basically serves to capture the stylized facts and allows us to use very simple tools to estimate the trend with a desired percentage of smoothness, as indicated below.

We are concerned here with the estimation of the trend component given the sample of observations $\{Z_t\}$ for $t = 1, \dots, N$ and means of PLS as did Guerrero (2008) in the univariate case. To do that we consider the problem of minimizing the following quadratic function

$$M(\lambda) = \sum_{t=1}^N (Z_t - \tau_t)' W (Z_t - \tau_t) \quad (2)$$

$$+ \lambda \sum_{t=3}^N \sum_{i=1}^2 (\tau_{i,t} - 2\tau_{i,t-1} + \tau_{i,t-2})^2$$

With $\lambda > 0$ a smoothing constant that penalizes the lack of trend smoothness and W a known symmetric and positive definite matrix of constant weights. Thus, for all t and $i = 1, 2$, if we let $\lambda \rightarrow 0$ both trends approach the observed data, i.e., $\tau_{i,t} \rightarrow Z_{i,t}$, and when $\lambda \rightarrow \infty$, $\tau_{i,t} - 2\tau_{i,t-1} + \tau_{i,t-2} \rightarrow 0$, so that every element of τ_t tends to behave like a straight line. Now, the smoothness of a trend depends only on its length (N), the value of the smoothing parameter (λ) and the correlation between cycles, as can be seen in the smoothness index (10) presented below. Note that only one smoothing constant is used to smooth both trends –unemployment and growth– since at the present situation both time series: (i) have the same length and (ii) share the same amount of smoothness.

Note also that there are some slight differences among different authors in the treatment of endpoints when using the HP filter. Some authors take the second summation similar to (2) to be over $t = 3$ to N , while others take the summation over $t = 2$ to $N - 1$, or some others take it over $t = 1$ to N . There is no clear preference. We chose to follow Guerrero's (2007) bivariate expression. This author addressed the end-of-sample problem within a PLS setting by centering the differenced trend series about a nonzero mean μ . The effect of including this parameter is to get a better fit of the trend at both ends of the original series, thus improving the forecasting ability of the trend. We do not include such a mean here because our interest in the present work is not to forecast the bivariate trend but to estimate Okun's

coefficient. If we consider the stacked vectors $Z = (Z'_1, \dots, Z'_N)'$ and $\tau = (\tau'_1, \dots, \tau'_N)'$, the problem can be posed as that of minimizing

$$M(\lambda) = (Z - \tau)'(I_N \otimes W)(Z - \tau) + \lambda \tau'(K'K \otimes I_2)\tau \quad (3)$$

where I_N is the N -dimensional identity matrix, \otimes denotes Kronecker product and K is an $(N - 2) \times N$ matrix representing the second order difference operator, that is,

$$K = \begin{pmatrix} 1 & -2 & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & \dots & 0 & 0 \\ \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \end{pmatrix} \quad (4)$$

The solution to the minimization of (3) is provided by Guerrero, Islas and Ramirez (2017, GIR from here on), that is

$$\hat{\tau} = (I_{2N} + \lambda K'K \otimes W^{-1})^{-1}Z \quad (5)$$

This solution provides the same numerical results as those obtained by Kalman filtering with smoothing, as shown by Gómez (1999) in the univariate case. It should be stressed that (5) is valid when both λ and W are known. Thus, the practical problem lies in providing adequate values for those parameters. GIR solved this problem by applying Generalized Least Squares (GLS) on the assumption that the smooth trend behavior is described by

$$\varepsilon_t = \tau_t - 2\tau_{t-1} + \tau_{t-2} \text{ for } t = 3, 4, \dots, N \quad (6)$$

and making $W = \Sigma_\eta^{-1}$ and $\lambda I_2 = \Sigma_\varepsilon^{-1}$, with Σ_n , the symmetric variance-covariance matrix of η_t and Σ_ε , the diagonal variance-covariance matrix of ε_t , that is

$$\Sigma_n = \begin{pmatrix} \sigma_{\eta_1}^2 & \sigma_{\eta_1\eta_2} \\ \sigma_{\eta_2\eta_1} & \sigma_{\eta_2}^2 \end{pmatrix} \text{ and } \Sigma_\varepsilon = \begin{pmatrix} \sigma_{\varepsilon_1}^2 & 0 \\ 0 & \sigma_{\varepsilon_2}^2 \end{pmatrix} \quad (7)$$

Then, the variance-covariance matrix of the GLS estimate is given by

$$Var(\hat{\tau}) = (I_2 \otimes \Sigma_\eta^{-1} + KK \otimes \Sigma_\varepsilon^{-1})^{-1} \quad (8)$$

where Σ_η and Σ_ε are assumed to be known. GIR showed that the correlation between trends may be taken into account by using a nondiagonal matrix Σ_ε , whose effect is practically irrelevant for trend estimation. In fact, this article showed that even correlations as high as 0.95 produce an increment in smoothness from 80% to slightly over 83% for sample sizes between 100 and 400, which covers the sample size considered in this paper.

2.1. Smoothing constant chosen by controlling smoothness

A feasible solution to the above problem is based on the idea that the smoothing constant should be calibrated rather than estimated, in order to avoid the need to validate the assumptions of the model underlying the estimation method employed. Thus, we suggest choosing the value of λ in such a way as to provide estimated trends with a percentage of smoothness chosen by the analyst *a priori*. The percentage of smoothness is related to the precision (the inverse of the variance-covariance matrix) of the estimated trend as follows. First, note that the precision of the trend estimate is given by the sum of the precision of the unobserved-component model (1) and the precision of the smoothness representation (6), that is,

$$\Gamma^{-1} = [Var(\hat{\tau})]^{-1} = I_2 \otimes \Sigma_\eta^{-1} + K'K \otimes \Sigma_\varepsilon^{-1} \quad (9)$$

Thus, the proportion of precision attributable to smoothness equals the proportion of the matrix $K'K \otimes \Sigma_\varepsilon^{-1}$ with respect to Γ^{-1} . A scalar measure of the proportion of P in $P + Q$, for the symmetric positive definite matrices P and Q was provided by Theil (1963) and is given by $\Delta(P; P + Q) = tr [P(P + Q)^{-1}] / N$. This is the only measure satisfying the following conditions: (i) it takes on values in (0; 1); (ii) it adds up to one in the sense that $\Delta(P; P + Q) + \Delta(Q; P + Q) = 1$; (iii) it is invariant under linear non-singular transformations of the variable involved, and (iv) it behaves linearly. See Theil (1963) for a proof. When the two matrices involved are precision matrices we refer to $\Delta(P; P + Q)$ as a measure of precision share of P in $P + Q$.

The measure of precision share is translated into a smoothness index by GIR where it is expressed as

$$\Delta(K'K \otimes \Sigma_\varepsilon^{-1}; \Gamma^{-1}) = 1 - \text{tr} \left[(I_{2N} + \lambda \beta \otimes K'K)^{-1} \right] / (2N) \quad (10)$$

with $\beta = \Sigma_n \text{Diag}(\sigma_{\eta_1}^{-2}, \sigma_{\eta_2}^2)$ the matrix of regression coefficients of one cycle on the other. A particularly important case arises when no correlation between cycles exists, in that case $\beta = I_2$ and $\Delta(K'K \otimes \Sigma_\varepsilon^{-1}; \Gamma^{-1})$ becomes the univariate index defined by Guerrero (2008).

Let us recall that no correlation is allowed between the two trends involved, which is a not too restrictive assumption, since ignoring that correlation causes a negligible amount of over-smoothness. The importance of the smoothness index lies in that we can fix at the outset of the study some desired percentage of smoothness to be achieved by the trend estimate and solve equation (10) for λ , given a preliminary estimate of the matrix β . As an aid to choosing an appropriate percentage of smoothness, GIR provided some guidelines that arise from a simulation study; and indicated how to obtain the required preliminary estimates. The Appendix presents the basic steps required to apply GIR's procedure, as well as an extract of the most important table.

2.2. Numerical estimation by Kalman filtering

To get $\hat{\tau}$ from (5), a $2N \times 2N$ matrix has to be inverted and this calculation may cause instability and an imprecise numerical solution when N is large. Thus, the penalized approach has the advantage of showing explicitly the role played by λ , but it does not provide an efficient calculation tool.

The estimation procedure for the proposed BHP filter is simplified if we rewrite the minimization problem underlying the filter in a state-space form. Casting the model in this form makes it possible to use the Kalman filter for parameter estimation. The representation considers equation (1) as a measurement equation, that is

$$\begin{bmatrix} Z_{y,t} \\ Z_{u,t} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \tau_{y,t} \\ \tau_{y,t-1} \\ \tau_{u,t} \\ \tau_{u,t-1} \end{bmatrix} + \begin{bmatrix} \eta_{y,t} \\ \eta_{u,t} \end{bmatrix} \quad (11)$$

where the sub-indices “y” and “u” correspond to output and unemployment, respectively. Then, we write (6) as the following State equation

$$\begin{bmatrix} \tau_{y,t} \\ \tau_{y,t-1} \\ \tau_{u,t} \\ \tau_{u,t-1} \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \tau_{y,t-1} \\ \tau_{y,t-2} \\ \tau_{u,t-1} \\ \tau_{u,t-2} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_{y,t} \\ \varepsilon_{u,t} \end{bmatrix} \quad (12)$$

and complement the representation with the variance-covariance matrices (7) written as

$$\text{Var} \begin{bmatrix} \eta_{y,t} \\ \eta_{u,t} \end{bmatrix} = \begin{bmatrix} \sigma_{\eta_y}^2 & \sigma_{\eta_y \eta_u} \\ \sigma_{\eta_u \eta_y} & \sigma_{\eta_u}^2 \end{bmatrix} \quad \text{and} \quad \text{Var} \begin{bmatrix} \varepsilon_{y,t} \\ \varepsilon_{u,t} \end{bmatrix} = \begin{bmatrix} \sigma_{\varepsilon_y}^2 & 0 \\ 0 & \sigma_{\varepsilon_u}^2 \end{bmatrix} \quad (13)$$

The parameter estimates in the above system are obtained by starting with an initial guess for the state vector and its covariance matrix. The Kalman filter then generates the prediction and updating equations recursively. Ultimately, the filter generates estimates of the unobserved components $\hat{\tau}_{y,t}$ and $\hat{\tau}_{u,t}$ as well as $\hat{\eta}_{y,t}$ and $\hat{\eta}_{u,t}$ for $t = 1, \dots, N$. To equate the results of the Kalman filter with smoothing, with those obtained directly from (5) we assume that $\sigma_{\varepsilon_y}^2 = \sigma_{\varepsilon_u}^2 = 1/\lambda$ and $\sigma_{\eta_y}^2 = \sigma_{\eta_u}^2 = 1$.

On the other hand, Okun (1962) suggested the existence of a strong link between the output gap and the unemployment gap, and exploited the tight correlation between these two components to predict the transitory component of output, given the transitory component of unemployment, by means of the following relation

$$Z_{y,t} - \tau_{y,t} = \theta (Z_{u,t} - \tau_{u,t}) + \delta_t \quad (14)$$

where θ represents Okun’s coefficient and δ_t is a random error with a mean of zero and constant variance. Then, as already pointed out, we estimate the two cyclical components jointly, together with

$$\hat{\theta} = \hat{\sigma}_{\eta_y \eta_u} / \hat{\sigma}_{\eta_u}^2 \quad (15)$$

in order to avoid the bias and inconsistency of the conventional OLS estimator obtained with a two-step procedure.

3. Empirical application

The dataset used in this application is the same one used by Sinclair (2009).¹ It contains quarterly data on real GDP in logs, $Z_{y,t}$, and civilian unemployment rate, $Z_{u,t}$, from 1948:1 up to 2005:4. Sinclair employed a theoretical bivariate correlated unobserved-component model and we have now the opportunity to contrast the results from that study with those produced by the BHP filter in terms of estimated components of output and unemployment, as well as the Okun's coefficient obtained by that study. We carried out all computations with the *WinRATS* package, version 8.10 (www.estima.com).

First of all, we followed the guidelines in GIR who recommend to visually appreciate the behavior of the two series under study; since in this case one of the series follows essentially a straight line (GDP in logs) and the other one does not behave as a straight line (unemployment), GIR's recommendation led us to the decision of using 80% smoothness for this application. The correlation $\tilde{\rho}_{\eta_y \eta_u} = -0.6$ was suggested by step b) of the estimating algorithm (see the appendix) so that the smoothing constant corresponding to 80% smoothness for the trend with $N = 232$, was obtained from table A in the appendix and became $\lambda = 16.29$. On the other hand, table 1 reports the estimates of the variance-covariance matrix Σ_n and $\rho_{\eta_y \eta_u}$ obtained from step c) of the estimating algorithm, while figures 1 and 2 show the estimated trend components of real GDP in logs and unemployment rate respectively, along with two-standard error limits derived from $\hat{\sigma}_{\eta_y}^2$ and $\hat{\sigma}_{\eta_u}^2$.

Table 1
Estimates of the variance-covariance matrix Σ_n and $\rho_{\eta_y \eta_u}$

| Parameter | $\sigma_{\eta_y}^2$ | $\sigma_{\eta_u}^2$ | $\sigma_{\eta_y \eta_u}$ | $\rho_{\eta_y \eta_u}$ |
|-----------|---------------------|---------------------|--------------------------|------------------------|
| Estimate | 0.9670 | 0.1893 | -0.3462 | -0.8093 |

We should stress that 80% smoothness was decided on by following suggestions for choosing appropriate percentages of smoothness

¹ Civilian unemployment rate data come from FRED II (Federal Reserve Economic Data) from the Federal Reserve Bank of St. Louis, available at the -July, 2006 revision-. The unemployment rate is the average unemployment rate over each quarter. Real GDP data from the US Department of Commerce, Bureau of Economic Analysis (BEA), based on the March 25, 2004 release.

in GIR, which depend on the type of behavior of the time series under study, as visually appreciated in time series plots. These suggestions include asking for high percentages of smoothness, say above 85%, when both trends follow a straight-line pattern. When one trend is linear and the other is nonlinear, the percentage of smoothness should diminish to between 80% and 85%; and when both trends are nonlinear it is reasonable to ask for lower amounts of smoothness, say below 80%. GIR concluded that the estimated trends with these choices resemble the “true” trends of the simulation reasonably well.

The two estimated trend components were produced by the Kalman filter with a smoother that uses all information available in the sample, thus providing a better “in-sample” fit as compared with the basic Kalman filter without smoothing. The cycle components, obtained by subtracting the corresponding trends from the observed series are plotted at the top of figures 1 and 2, together with those obtained by Sinclair (2009), where the shading in those figures represents NBER recession dates.

As in Sinclair (2009), the estimated permanent components of GDP and unemployment are not as smooth as those obtained within the framework of business cycle theory. In fact, the estimated permanent component of GDP looks slightly more variable than the series itself. This result is similar to that obtained by the Beveridge-Nelson decomposition of GDP series and in Morley, Nelson, and Zivot (2003); but it is contrary to Clark (1989) which separately decomposes output and unemployment into trend and stationary components. He used the covariance between the stationary components of the two series to support his case for the existence of large temporary movements in real output.

Clark (1989) found evidence that the correlation between the permanent and transitory components of GDP is statistically non-significant. On the other hand, when the components of GDP and unemployment are jointly estimated with no restriction on the covariances, as in Sinclair (2009), the results are significantly different from Clark (1989), and provide further support of a small cyclical component for GDP. Even more, as suggested by Dupasquier, Guay and St-Amant (1999), if the model includes the dynamics of permanent shocks in potential output, the size of the output gap would be extremely small.

Figure 1
U.S. unemployment and estimated components, 1948:1 2005:4

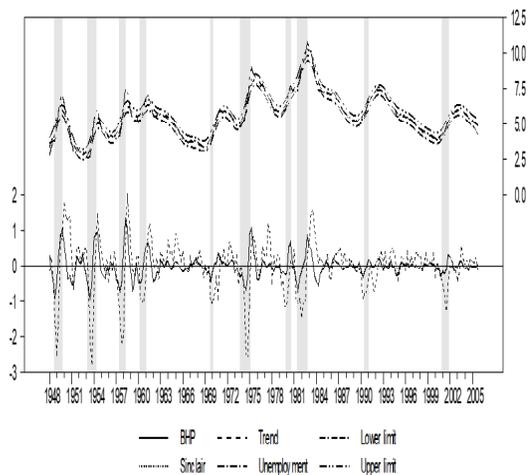
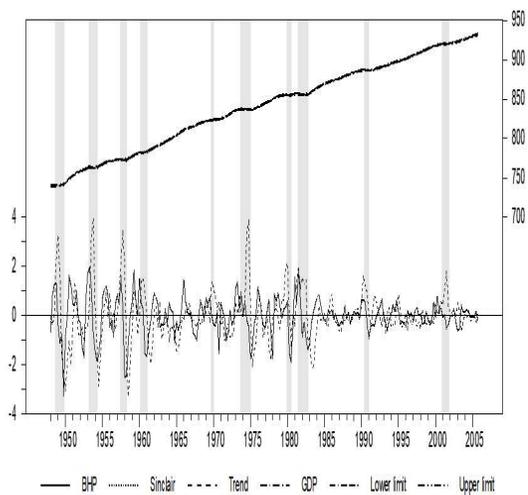
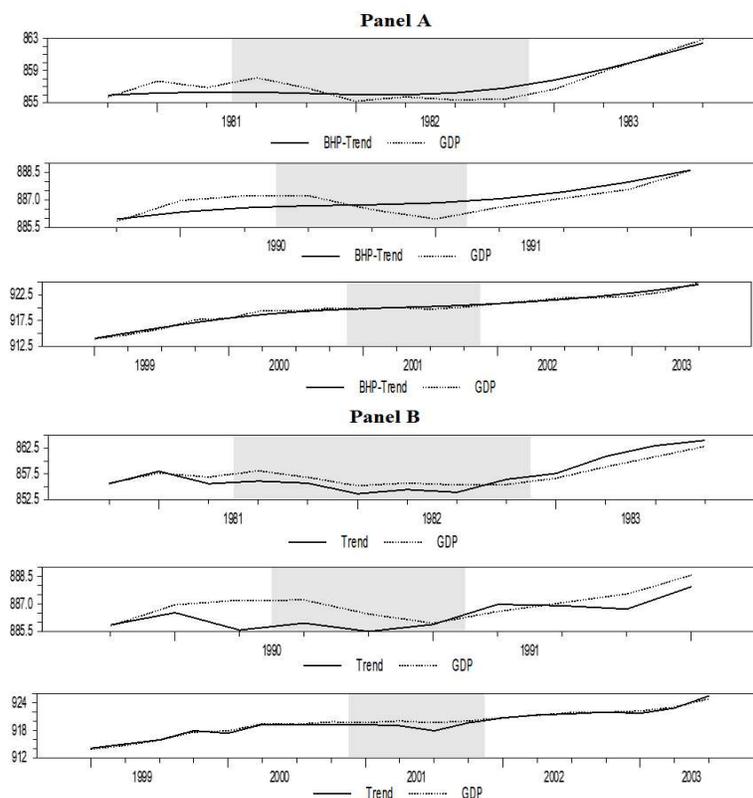


Figure 2
U.S. real GDP and estimated components, 1948:1 2005:4



A closer examination of figures 1 and 2 indicates not only that the transitory shocks of our estimates display comparable magnitudes with respect to the ones obtained by Sinclair (2009), but also that the turning points in our estimates indicate a little more clearly the phases of the economic cycle. The turning points in Sinclair seem to be delayed by two periods because of the Auto-Regressive structure of order 2 in the model employed. In figure 3, a close-up of the 1981, 1990 and 2001 recessions, we can observe, as in Sinclair (2009), that the estimated permanent component dips well below the series in all three recessions. This implies a positive transitory component, since the series is above its steady state value, therefore our results differ from theories that describe recessions as temporary negative movements.

Figure 3
Close-up of the 1981, 1990 and 2001 recessions
Panel A: BHP's results, Panel B: Sinclair (2009) results



Regarding the permanent component of unemployment, as we found with GDP, a sizable amount of movement in the unemployment rate appears to arise from permanent shocks. A closer examination of figure 1 shows a positive permanent movement in unemployment, rather than the temporary movements that some theories of recession would predict. Similar to Sinclair’s proposed model, the BHP filter also provides new estimates of the relative importance of permanent versus transitory movements in the unemployment rate.

Many models have previously treated unemployment as stationary, including Blanchard and Quah (1989), who identify supply disturbances as those that have a permanent effect on output, and demand disturbances as those that have no permanent effect on output. These authors find a tight relation for demand disturbances, but supply disturbances present a confusing picture. Some authors (*e.g.* Sinclair, 2009) have explained that this is due to the fact that their models don’t allow for a permanent stochastic component to unemployment. Furthermore, Blanchard and Quah’s assumption that all movements in unemployment are temporary drives the finding of a much large transitory component.

The results of Blanchard and Quah (1989), and Clark (1989) are appealing because the estimated transitory components appear to follow the NBER business cycle. In particular, the transitory component of GDP in Clark is similar to the cycle obtained when a deterministic linear trend is used to represent the permanent component of GDP. Even though both of these transitory components match those found in textbooks, they are rejected by the data when less restrictive models are used, as in Sinclair (2009) and GIR. As discussed by Nelson (1988), it appears that the estimates of the transitory component are actually spurious cycles, not the properly estimated transitory component. We want to point out that the methodology used in this paper was tested through simulation to avoid this error, and our results show that more complex model specifications, even though they may seem more appropriate, may not yield better results than those produced here.

The results presented in this work support those of Sinclair, 2009, and, likewise, call into question a number of macroeconomic theories for the post-war US. In particular, studying permanent movements separately from transitory movements in GDP or unemployment implicitly requires the assumption that they are uncorrelated. Furthermore, “...*theories explaining only growth (permanent movements) or only transitory movements cannot provide adequate macroeconomic insights if there are important interactions between the two...*” Stock

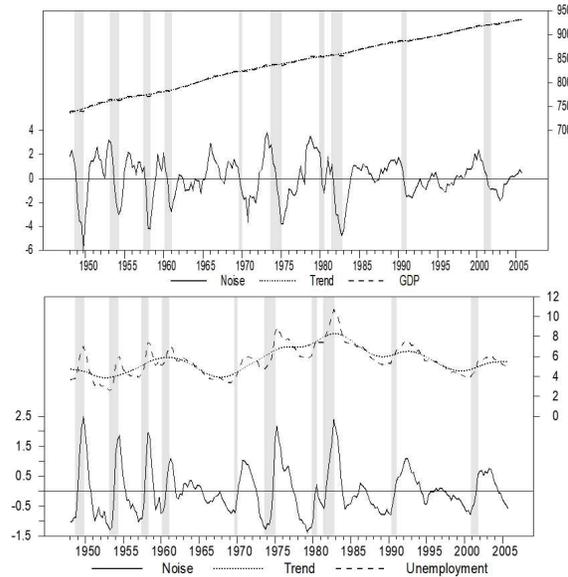
and Watson (1988: 148). Therefore, the results of this paper, as well as those of Sinclair, cast doubt on models that study GDP growth and transitory movements separately, as well as on models that treat business cycle movements as exclusively temporary, and theories such as the often-called models of hysteresis, where some part of the temporary shock persists and becomes permanent. On the other hand, our results are in line with real business cycle theories such as the one presented by Kydland and Prescott (1982) and Prescott (1987), where transitory movements in the series arise primarily from adjustments to permanent changes.

Okun's law, given by equation (14), suggests that the transitory components of output and unemployment rate are negatively correlated. This finding is relevant to answering the following question: given a certain level of unemployment, what level of GDP should one expect under the economic conditions prevailing during the sample period? To answer this question, we estimated Okun's coefficient by means of equation (15) and obtained $\hat{\theta} = -0.3462/0.1893 = -1.82$, implying that a 1 percentage-point decrease in transitory unemployment corresponds to a 1.82 percentage-point increase in transitory real GDP. Even though both our estimate and that of Sinclair (2009), -1.4, are below the 2% consensus estimate (*e.g.* Grant, 2002), ours is closer and both are within the range of usual estimates, which varies between 3% (Okun, 1962) and 0.67% (Prachowny, 1993).

We then compare Okun's coefficient estimated using the BHP filter with the estimate obtained through the conventional two-step procedure; in the first step, the permanent components of the output and unemployment series were estimated separately using the univariate HP filter with the smoothing parameter set to 1600, as is standard for quarterly data. This λ value produces a smoother trend than the one suggested by our proposal ($\lambda = 16.29$). This smoother trend incorrectly includes permanent movements in the transitory components of GDP and unemployment that lead to an upward bias in the estimation of Okun's coefficient in the second step, which applies OLS to (14) and takes on the value $\hat{\theta} = -1.9$.

Figure 4 allows us to appreciate the results of the HP filter as compared with those produced by the BHP filter and Sinclair (2009). We should be aware that the smoothness achieved varies according to the sample size. For instance, the smoothness achieved by applying the HP filter to the GDP and the unemployment rate data with $N = 232$ and $\lambda = 1600$ is about 93.9%. Therefore, ignoring the correlation between noise components leads to smoother trends and more incorrect inclusion of persistent noise components.

Figure 4
Real GDP, the unemployment rate and the estimated components produced by the univariate HP filter with $\lambda = 1600$



4. Conclusions

The suggested BHP filter is an easy-to-use tool that can be employed to estimate trends of bivariate time series with controlled smoothness, that is, when we want to establish (valid) comparisons between trends obtained with the same pre-specified percentage of smoothness. In addition, comparisons between the corresponding cycles are also valid. Of course, univariate time series filtering is even easier to apply but it may give misleading results since it does not take simultaneity in the estimation into account. On the other hand, we showed that some more complex model specifications, even though appropriate, may not yield better results than those produced by the BHP filter. Furthermore, we should keep in mind that the results produced by a filter do not rely heavily on statistical assumptions, and such assumptions, which are the basis for more complex theoretical models, must be validated with the data at hand.

In the empirical application shown here we found that the results obtained with the BHP filter reproduce the most important, and pos-

itive, features of a more complicated procedure based on a structural model and avoid a negative aspect related to the displacement of the cycle phases. Moreover, as compared with the univariate HP filter, we found that the BHP filter produces an estimate of the correlation coefficient that avoids the bias and inconsistency of the usual two-step procedure associated with the HP filter.

Acknowledgements

The authors gratefully acknowledge the comments and suggestion of two anonymous referees and the editor of this journal. They also thank Asociación Mexicana de Cultura, A.C. for providing financial support to carry out this work. Alejandro acknowledges a sabbatical scholarship from the National Council for Science and Technology of Mexico (Conacyt). This work was partially done whilst he was visiting the Department of Economics, Nepal Study Center, and the RWJF Center for Health Policy at the University of New Mexico, USA.

References

- Adams, C. and D.T. Coe. 1990. A systems approach to estimating the natural rate of unemployment and potential output for the United States, *IMF Staff Papers*, 37: 232-293.
- Blanchard, O.J. and D. Quah, 1989. The dynamic effects of aggregate demand and supply disturbances, *The American Economic Review*, 79(4): 655-673.
- Boone, L. 2000. Comparing semi-structural methods to estimate unobserved variables: The HPMV and Kalman filters approaches, OECD Economics Department Working Papers, no. 240.
- Chagny, O. and M. Lemoine, 2002. The impact of the macroeconomic hypothesis on the estimation of the output gap using a multivariate Hodrick-Prescott filter: The case of the Euro area, Eurostat Colloquium on Modern Tools for Business Cycle Analysis.
- Clark, P. K. 1987. The cyclical component of U.S. economic activity, *The Quarterly Journal of Economics*, 102: 797-814.
- . 1989. Trend reversion in real output and unemployment, *Journal of Econometrics*, 40: 15-32, DOI: 10.1016/0304-4076(89)90027-4.
- Dermoune, A., B. Djehiche, and N. Rahmania. 2009. Multivariate extension of the Hodrick-Prescott Filter-Optimality and characterization, *Studies in Nonlinear Dynamics & Econometrics*, 13(3): 1-33.

- Dupasquier, C., A. Guay, and P. St-Amant. 1999. A survey of alternative methodologies for estimating potential output and the output gap, *Journal of Macroeconomics*, 21(3): 577-595.
- Evans, G.W. 1989. Output and unemployment dynamics in the United States: 1950-1985, *Journal of Applied Econometrics*, 4(3): 213-237, DOI: 10.1002/jae.3950040302.
- Federal Reserve Bank of St. Louis. 2006. FRED II (Federal Reserve Economic Data), <<https://fred.stlouisfed.org/>>.
- Ford, R. and D.E. Rose. 1989. Estimates of the *NAIRU* using an extended Okun's Law, Working Paper, no. 89-3, Bank of Canada.
- Gómez, V. 1999. Three equivalent methods for filtering finite nonstationary time series, *Journal of Business and Economic Statistics*, 17: 109-166.
- Grant, A.P. 2002. Time-varying estimates of the natural rate of unemployment: A revisit of Okun's Law, *The Quarterly Review of Economics and Finance*, 42(1): 95-113, DOI: 10.1016/S1062-9769(01)00109-0.
- Guerrero, V.M. 2007. Time series smoothing by penalized least squares, *Statistics and Probability Letters*, 77: 1225-1234, DOI: 10.1016/j.spl.2007.03.006.
- . 2008. Estimating trends with percentage of smoothness chosen by the user, *International Statistical Review*, 76(2): 187-202, DOI: 10.1111/j.1751-5823.2008.00047.x.
- , C.A. Islas, and R.L. Ramirez. 2017. Trend estimation of multivariate time series with controlled smoothness, *Communications in Statistics-Theory and Methods*, 46(13): 6704-6726, DOI: 10.1080/03610926.2015.1133826.
- Hodrick, R.J. and E.C. Prescott. 1997. Postwar U.S. business cycles: An empirical investigation, *Journal of Money Credit and Banking*, 29: 1-16.
- Kydland, F.E. and E.C. Prescott. 1982. Time to build and aggregate fluctuations, *Econometrica*, 50(6): 1345-1370.
- Laxton, D. and R. Tetlow. 1992. A simple multivariate filter for the measurement of potential output, Technical Report no. 59, Bank of Canada.
- Morley, J.C., C.R. Nelson, and E. Zivot. 2003. Why are the Beveridge-Nelson and unobserved-components decompositions of GDP so different? *The Review of Economics and Statistics*, 85(2): 235-243.
- Nelson, C.R. 1988. Spurious trend and cycle in the state decomposition of a time series with a unit root, *Journal of Economic Dynamics and Control*, 12(2/3): 475-488.
- , and C.I. Plosser. 1982. Trends and random walks in macroeconomic time series: Some evidence and implications, *Journal of Monetary Economics*, 10: 139-162.
- Okun, A.M. 1962. Potential GNP: Its measurement and significance, American Statistical Association, *Proceedings of the Business and Economic Statistics Section*, 98-104.
- Prachowny, M.F.J. 1993. Okun's law: Theoretical foundations and revised estimates, *Review of Economics and Statistics*, 75: 331-336, DOI: 10.2307/2109440.
- Prescott, E.C. 1987. Theory ahead of business cycle measurement, *Carnegie-Rochester Conference Series on Public Policy*, 25: 11-44.

- Sinclair, T.M. 2009. The relationship between permanent and transitory movements in U.S. output and the unemployment rate, *Journal of Money, Credit and Banking*, 41(2-3): 529-542.
- Stock, J.H. and M.W. Watson. 1988. Variable trends in economic time series, *Journal of Economic Perspectives*, 2: 147-174.
- Theil, H. 1963. On the use of incomplete prior information in regression analysis, *Journal of the American Statistical Association*, 58: 401-414, DOI: 10.2307/2283275.
- US Department of Commerce. 2004. Bureau of Economic Analysis (BEA), <<http://www.bea.doc.gov/bea/dn/home/gdp.htm>>.

Appendix: Summary of GIR's procedure

We present here a brief description of GIR's algorithm and the most important table used to estimate bivariate trends.

A.1. Algorithm

Step a) Compute preliminary estimates of Σ_n and Σ_ε as in proposition 2 in GIR, that is,

$$\tilde{\Sigma}_\eta = \frac{1}{34} \left[\hat{\gamma}(2) + \hat{\gamma}'(2) - 4\hat{\gamma}(1) - 4\hat{\gamma}'(1) \right]$$

and

$$\tilde{\Sigma}_\varepsilon = \hat{\gamma}(0) - \frac{3}{17} \left[\hat{\gamma}(2) + \hat{\gamma}'(2) - 4\hat{\gamma}(1) - 4\hat{\gamma}'(1) \right]$$

where $\hat{\gamma}(0)$, $\hat{\gamma}(1)$, and $\hat{\gamma}(2)$ are estimated cross-correlation matrices of the twice-differenced bivariate time series while $\hat{\gamma}'(1)$, and $\hat{\gamma}'(2)$, are their transposed matrices.

Step b) Let the data suggest a value for the smoothing parameter, that is, calculate λ as the arithmetic average of the estimates of $\sigma_{\eta_y}^2/\sigma_{\varepsilon_y}^2$ and $\sigma_{\eta_u}^2/\sigma_{\varepsilon_u}^2$, where $\sigma_{\eta_y}^2$ and $\sigma_{\eta_u}^2$ are elements of Σ_n , while $\sigma_{\varepsilon_y}^2$ and $\sigma_{\varepsilon_u}^2$ are elements of Σ_ε

Step c) Preliminarily estimate the trend vector

$$\tilde{\tau} = (\mathbf{I}_{2N} + \tilde{\lambda} \mathbf{K}'\mathbf{K} \otimes \mathbf{I}_2)^{-1} \mathbf{Z}$$

its MSE matrix

$$\tilde{\Sigma}_\eta = \frac{1}{N-2} \left[\sum_{t=1}^N (\mathbf{Z}_t - \tilde{\tau}_t) (\mathbf{Z}_t - \tilde{\tau}_t)' + \tilde{\lambda} \sum_{t=3}^N (\nabla^2 \tilde{\tau}_t) (\nabla^2 \tilde{\tau}_t)' \right]$$

and the between-cycle correlation

$$\tilde{\rho}_{\eta_y \eta_u} = \tilde{\sigma}_{\eta_y \eta_u} / (\tilde{\sigma}_{\eta_y} \tilde{\sigma}_{\eta_u})$$

Step *d*) With the values of N and $\tilde{\rho}_{\eta_y \eta_u}$, choose λ from table 3 (see the extract below) so that a desired percentage of smoothness can be achieved for the trend.

Step *e*) Compute the final estimates of the bivariate trend $\hat{\tau}$ as in (5) and $Var(\hat{\tau})$ as in (8) or else by applying Kalman filtering, on the assumption that $\sigma_{\varepsilon_y}^2 = \sigma_{\varepsilon_u}^2 = 1/\lambda$ and $\sigma_{\eta_y}^2 = \sigma_{\eta_u}^2 = 1$, with the value for λ chosen in step *d*).

A.2. *Extract of table 3 in GIR*

Table A

Smoothing parameter λ for some choices of sample size (N) and between-cycle correlation (ρ) for 80% smoothness

| <i>Sample Size (N)</i> | <i>Between-cycle correlation (ρ)</i> | | | | |
|-------------------------------------|--|-------|-------|-------|-----|
| | ... | 0.4 | 0.6 | 0.8 | ... |
| 224 | ... | 13.62 | 16.29 | 24.05 | ... |
| 256 | ... | 13.48 | 16.12 | 23.81 | ... |