

PREFERENCES FOR SHIFTS IN PROBABILITIES AND EXPECTED UTILITY THEORY*

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Resumen: Las preferencias definidas sobre loterías inducen preferencias sobre cambios en probabilidades. Definimos el axioma de independencia para preferencias definidas sobre cambios en probabilidades. Al usar esta versión del axioma de independencia construimos una función de utilidad esperada. Finalmente, desarrollamos la geometría aditiva correspondiente.

Abstract: We translate preferences over lotteries into preferences over “shifts in probabilities”. We define the independence axiom for such preferences. Next we construct and expected utility function to represent the preferences and present the corresponding additive geometry.

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1. Motivation

Consider the following situation. Initially, an agent faces an even lottery: with probability $\frac{1}{2}$ she wins 5 pesos and with probability $\frac{1}{2}$ she wins 10 pesos. Consider a shift in probability of $\frac{1}{3}$ from outcome “winning 5 pesos” to outcome “winning 6 pesos”:

<i>Outcomes</i>	<i>1st Lottery</i>	<i>Shift in probability</i>	<i>2nd Lottery</i>
5	$\frac{1}{2}$	$-\frac{1}{3}$	$\frac{1}{6}$
6	0	$+\frac{1}{3}$	$\frac{1}{3}$
7	0	-	0
8	0	-	0
9	0	-	0
10	$\frac{1}{2}$	-	$\frac{1}{2}$

Which shift in probability from the second lottery might lead to a third lottery indifferent to the first one. For some preferences, an answer might be a shift of $\frac{1}{4}$ from outcome “winning 10 pesos” to outcome “winning 8 pesos”:

<i>Outcomes</i>	<i>1st Lottery</i>	<i>Shift in probability</i>	<i>2nd Lottery</i>
5	$\frac{1}{6}$	-	$\frac{1}{6}$
6	$\frac{1}{3}$	-	$\frac{1}{3}$
7	0	-	0
8	0	$+\frac{1}{4}$	0
9	0	-	0
10	$\frac{1}{2}$	$-\frac{1}{4}$	$\frac{1}{4}$

Thus, from the initial lottery, the first shift in probability is compensated, thus comparable with the second shift in probability, although 5, 6, 8 nor 10 are indifferent outcomes. Actually, even if the answer should be “none”, the question makes sense.

Any preferences over lotteries induces preferences over shifts in probabilities. We present a complementary approach to expected utility theory¹ by presenting a characterization of expected utility

¹ References on the subject are, for instance, Hammond (1998), Kreps (1988), and Wakker (1988).

where the independence axiom deals with preferences defined over shifts in probabilities.

The relevance of our approach is three-fold. First it makes it clear that the cardinal nature of expected utility is only a behavioral characteristic rooted in the independence axiom and has no normative appeal. Second, our proof is entirely constructive, which provides insight into the way the independence axiom leads to expected utility. Third, we deal strictly with preferences defined over simple lotteries.

Moreover, applying Thales’s Theorem, we relate the fact that indifference “curves” are straight lines with weighted additive geometry in the Marshak-Machina triangle, thus providing a geometric interpretation of expected utility.

2. Preferences Over Shifts in Probabilities

Let $C = \{a_1, \dots, a_n\}$ be the set of finite outcomes and L a lottery defined over C . Given a lottery L which assigns to event a a probability larger than α and to event b a probability smaller than $1 - \alpha$, $\alpha \in [0, 1]$, $L(\alpha\Delta_a^b)$ is the lottery where the probability assigned to outcome a has been lowered by an amount of α and the probability assigned to outcome b has increased by an amount of α . For instance, if $C = (a, b, c, d, e)$, $L = (.1; .4; .3; 0; .2)$ and $\alpha = .1$, then $L(\alpha\Delta_a^b) = (0; .5; .3; 0; .2)$. The concatenation of the operation is denoted by $L(\alpha\Delta_a^b + \dots + \alpha'\Delta_{a'}^{b'})$.

The space of preferences over simple lotteries is denoted by \mathcal{L} , a preference relation in \mathcal{L} is given by \succsim . Given a lottery L , let p_a (p_b) be the probability of a (b) in the lottery L , and $D_L = \{(\alpha, a, b) \in [0, 1] * C * C \mid \alpha \leq \min\{p_a, 1 - p_b\}\}$. Consider a preference relation \succsim in \mathcal{L} , we define the induced preference relation over shifts in probabilities \succsim^Δ as follows: for any L and (α, a, b) in D_L

$$[\alpha\Delta_a^b \succsim^\Delta \beta\Delta_c^d \text{ for } L] \text{ if and only if } [L(\alpha\Delta_a^b) \succsim L(\beta\Delta_c^d)].$$

If the binary relation \succsim is complete and transitive, so is \succsim^Δ . To see it, think of the contrapositive statement.

These are heavy notations. They emphasize the richness of preferences over lotteries and how restrictive the independence axiom is. It says that the ordering of shifts in probabilities is independent of its weight and the original lottery.

DEFINITION 1. *The preference relation \succsim^Δ satisfies the independence axiom if, for all L , (α, a, b) and (β, c, d) in \mathcal{D}_L*

$$\alpha\Delta_a^b \succsim^\Delta \beta\Delta_c^d \text{ for } L \text{ if and only if } (\gamma\alpha)\Delta_a^b \succsim^\Delta (\gamma\beta)\Delta_c^d$$

for all L' and $\gamma > 0$ where $\gamma\alpha$ and $\gamma\beta$ belong to D_L and $D_{L'}$.

The equivalence holds with strict preferences or indifference. Consider the concatenation such that

$$L(\alpha\Delta_a^b + \beta\Delta_c^d) \sim L$$

It tells us in which proportion shifts in probabilities from a to b and from c to d keep the agent indifferent. By independence this marginal rate of substitution, $\frac{\alpha}{\beta}$, is constant (independent of the original lottery and of the size of the shift). Quoting Machina (1987), Hammond (1998) discusses how ratios of utility differences are equal to the marginal rates of substitution between probability shifts, which are constant when indifference curves are linear. Although it is somehow indirect to capture a property about preferences by dealing with the utility representation, it is an illuminating way to explain the cardinal nature of expected utility. Thus, we know that indifference curves are indeed straight lines. In our focus, in contrast, the constant marginal rate of substitution between probability shifts is a restriction put on the preferences over shifts in probabilities. It is clearly a choice behavior pattern, thus, our approach agrees with Weymark (2005) in the sense that cardinality has no normative appeal.

We also need a continuity condition to state our theorem.

DEFINITION 2. *The preference relation \succsim is continuous if for any L , L' and final outcomes a and b*

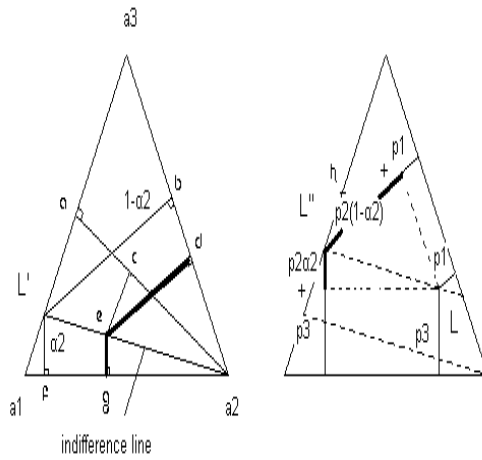
$$\{\alpha \in [0, 1] : L(\alpha\Delta_a^b) \succsim L'\} \text{ and } \{\alpha \in [0, 1] : L' \succsim L(\alpha\Delta_a^b)\}$$

are closed.

THEOREM 3. *Continuous preferences \succsim in \mathcal{L} are represented by an expected utility function if and only if induced preferences \succsim^Δ satisfy the independence axiom. Moreover, functions obtained from positive affine transformations of the utility function also represent these preferences.*

The proof of the theorem is entirely constructive, which allows to us exhibit, both geometrically and analytically, how the independence axiom leads to expected utility.²

Figure 1
Additive shifts in probability



PROOF. The complete proof is in the appendix. There, we adapt usual arguments to prove that:

1. There exists a worst lottery, \underline{L} , which is a degenerated lottery and assigns probability 1 to event, say, a_1 ,
2. There exists a preferred lottery, \overline{L} , which is degenerated and assigns probability 1 to event, say, a_n ,
3. For all lotteries L , there exists a lottery $\underline{L}(\alpha_L \Delta_{a_1}^{a_n})$ indifferent to L , and the higher α_L is, the greater the preference for L .

Thus, assigning to all lotteries L in \mathcal{L} the utility $u_L \equiv \alpha_L$ is always feasible and allows us to represent preferences.

Then $u_{a_1} \equiv 0$, $u_{a_n} \equiv 1$ and for all degenerated lotteries L which assign probability 1 to a_i , let $u_{a_i} \equiv \alpha_i$ where $\underline{L}(\alpha_i \Delta_{a_1}^{a_n}) \sim L$.

² We do not need to prove that the function which represents the preferences is linear and rely on the result which states that a utility function is linear if and only if it has an expected utility form.

Moreover, since $\underline{L}(\alpha_i \Delta_{a_1}^{a_n}) \sim L$, the shift in probability from $L = (0, \dots, 1, \dots, 0)$ to $\underline{L}(\alpha_i \Delta_{a_1}^{a_n}) = (1 - \alpha_i, \dots, 0, \dots, \alpha_i)$ indicates that the constant rate of substitution between probability shifts which keeps the agent indifferent with respect to L , is the following: for any lottery, if the probability assigned to a_i ($i \neq 1, n$) is lowered by p_i , to compensate the agent the probability assigned to a_n has to be increased by $p_i * \alpha_i$ and the probability assigned to a_1 has to be increased by $p_i * (1 - \alpha_i)$.

Figure 1 displays the geometry of the analysis in the Marshak-Machina triangle when $n = 3$, where $\underline{L}(\alpha_2 \Delta_{a_1}^{a_3})$ is denoted by L' . The indifference line of the left hand triangle exhibits the constant rate of substitution. By construction $a2a = 1$, and we choose c such that $a2c = p_2$ ($= Lh$ in the right hand triangle), $L'f = \alpha_2$ and $L'b = 1 - \alpha_2$. By Thales' Theorem:

$$\frac{a2c}{a2a} = \frac{a2e}{a2L'} \quad (\text{in triangle } a2, a, L'),$$

$$\frac{a2e}{a2L'} = \frac{eg}{L'f} \quad (\text{in triangle } a2, L', f),$$

$$\frac{a2c}{a2L'} = \frac{de}{bL'} \quad (\text{in triangle } a2, b, L').$$

Thus $eg = p_2\alpha_2$ and $de = p_2(1 - \alpha_2)$.

The right hand triangle exhibits the additive nature of the shifts in probabilities. It is the geometric interpretation of the expected utility form of the function which represents the preferences, since $LL'' = p_3 + p_2\alpha_2 = p_3u_3 + p_2u_2 + p_1u_1$.³

We now see analytically how these constant shifts in probability is performed by the independence axiom. Moreover, it is performed additively for a_2, \dots, a_{n-1} , which explains the expected utility form of the utility function.

We consider non degenerated-lotteries L and compute α_L . All lotteries $L = (p_1, \dots, p_n) \in \mathcal{L}$ can be expressed as the concatenation

$$\underline{L}(p_2 \Delta_{a_1}^{a_2} + \dots + p_n \Delta_{a_1}^{a_n}) = L.$$

Since by the definition of α_2 , $1 \Delta_{a_1}^{a_2} \sim \alpha_2 \Delta_{a_1}^{a_n}$ for \underline{L} , independence implies

³ We do not need to deal with induced preferences to develop the geometric interpretation.

$$\underline{L}(p_2\Delta_{a_1}^{a_2} + \dots + p_n\Delta_{a_1}^{a_n}) \sim \underline{L}(p_2\alpha_2\Delta_{a_1}^{a_n} + p_3\Delta_{a_1}^{a_3} + \dots + p_n\Delta_{a_1}^{a_n}).$$

Repeating the argument for $i = 3, \dots, n - 1$ leads to

$$\begin{aligned} &\underline{L}(p_2\alpha_2\Delta_{a_1}^{a_n} + p_3\Delta_{a_1}^{a_3} \dots + p_n\Delta_{a_1}^{a_n}) \\ &\sim \underline{L}(p_2\alpha_2\Delta_{a_1}^{a_n} + p_3\alpha_3\Delta_{a_1}^{a_n} + \dots + p_n\Delta_{a_1}^{a_n}) \end{aligned}$$

...

$$L = \underline{L}(p_2\Delta_{a_1}^{a_2} + \dots + p_n\Delta_{a_1}^{a_n}) \sim \underline{L}((p_2\alpha_2 + \dots + p_n)\Delta_{a_1}^{a_n}).$$

Since $\alpha_n = 1$, the probability distribution of $\underline{L}((p_2\alpha_2 + \dots + p_n)\Delta_{a_1}^{a_n})$ is

$$(1 - \sum_{i=2}^n (p_i\alpha_i), 0, \dots, 0, \sum_{i=2}^n (p_i\alpha_i)).$$

Remember that $u_{a_1} = \alpha_1 = 0$ and $u_{a_n} = \alpha_n = 1$. Hence the utility associated with L is $u_L \equiv \alpha_L = \sum_{i=2}^n (p_i\alpha_i) = \sum_{i=1}^n (p_i\alpha_i)$; namely

$$u_L = \sum_{i=1}^n (p_i u_i).$$

This utility function has the expected utility form. Obviously all positive affine transformations also represent the preferences. ■

3. Concluding Remarks

Now, the set of final outcomes C is a continuous interval in \mathfrak{R} . A lottery L is a probability distribution over C . The space of compound lotteries is \mathcal{L}^c . Suppose that L has the same probability distribution as $\pi_1 L_1 + \pi_2 L_2 + \dots + \pi_l L_l$, a compound lottery over simple lotteries L_1, L_2, \dots and L_l in \mathcal{L}^c . Then, $L(\alpha \Delta_{L_1}^{L_2})$ is the lottery where the weight given to L_1 has been lowered by $\alpha \leq \pi$ and the weight given to L_2 has been increased by α .

Given a lottery L , let p_1 (p_2) be the probability of L_1 (L_2) in the lottery L , and

$$D_L = \{(\alpha, L_1, L_2) \in [0, 1] * C * C \mid \alpha \leq \min \{p_1, 1 - p_2\}\}.$$

For any (α, L_1, L_2) in D_L , we define the induced preference relation \succsim^Δ as follows

$$[\alpha \Delta_{L_1}^{L_2} \succ^\Delta \beta \Delta_{L_3}^{L_4} \text{ for } L] \text{ if and only if } [L(\alpha \Delta_{L_1}^{L_2}) \succsim L(\beta \Delta_{L_3}^{L_4})].$$

The independence axiom is rephrased.

DEFINITION 4. *The preference relation \succsim^Δ satisfies the independence axiom if for all L and (α, L_1, L_2) in D_L*

$$\alpha \Delta_{L_1}^{L_2} \succ^\Delta \beta \Delta_{L_3}^{L_4} \text{ for } L \text{ if and only if } (\gamma \alpha) \Delta_{L_1}^{L_2} \succ^\Delta (\gamma \beta) \Delta_{L_3}^{L_4}$$

for all L' and γ where $\gamma \alpha$ and $\gamma \beta$ are in D_L and $D_{L'}$.

The independence axiom states that

$$\alpha \Delta_L^{L_1} \succ^\Delta \alpha \Delta_L^{L_2} \text{ for } L \text{ if and only if } (\gamma \alpha) \Delta_L^{L_1} \succ^\Delta (\gamma \alpha) \Delta_L^{L_2}$$

for all L' and γ well defined. When $\gamma \alpha = 1$ and $L = L'$, the reader has recognized the usual (strong) independence axiom defined on preferences over lotteries. Now, if these preferences are also continuous, they are represented by an expected utility function. Finally, when preferences are represented by an expected utility function, it is easy to check, as in the first part of the proof of our theorem, that the independence axiom for induced preferences (definition 4) holds. Thus, our approach is consistent with the approach of Von Neuman and Morgenstern (1944) when dealing with compound lotteries.

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Appendix

PROOF.(\Rightarrow) Suppose that all preferences \succsim in \mathcal{L} are represented by an expected utility function $u(\cdot)$. Thus, for all a_i in C , there exists a finite real number u_i such that $u(L) = \sum_{i=1}^n p_i u_i$ for all $L = (p_1, \dots, p_n)$. Without loss of generality, since all functions reached from affine transformation of the utility function also represent preferences, a_1, \dots, a_n are positive real numbers. Thus, when $0 \leq \alpha \leq p_l \leq 1$ and $0 \leq \beta \leq p_j \leq 1$,

$$u(L(\alpha \Delta_{a_l}^{a_m})) = p_1 u_1 + \dots + (p_l - \alpha) u_l + \dots + (p_m + \alpha) u_m + \dots + p_n u_n,$$

and

$$u(L(\beta \Delta_{a_j}^{a_k})) = p_1 u_1 + \dots + (p_j - \beta) u_j + \dots + (p_k + \beta) u_k + \dots + p_n u_n.$$

Suppose $\alpha \Delta_{a_l}^{a_m} \succsim^{\Delta} \beta \Delta_{a_j}^{a_k}$ for L , i.e., $u(L(\alpha \Delta_{a_l}^{a_m})) \geq u(L(\beta \Delta_{a_j}^{a_k}))$, namely

$$\begin{aligned} u(L(\alpha \Delta_{a_l}^{a_m})) - u(L(\beta \Delta_{a_j}^{a_k})) &\geq 0 \Leftrightarrow -\alpha u_l + \alpha u_m + \beta u_j - \beta u_k \geq 0, \\ &\Leftrightarrow \gamma(-\alpha u_l + \alpha u_m + \beta u_j - \beta u_k) \geq 0 \text{ for all,} \\ &\gamma > 0 \text{ independently of the original lottery.} \end{aligned}$$

Thus, whenever $L'(\gamma \alpha \Delta_{a_l}^{a_m})$ and $L'(\gamma \beta \Delta_{a_j}^{a_k})$ are well defined, $\gamma \alpha \Delta_{a_l}^{a_m} \succsim^{\Delta} \gamma \beta \Delta_{a_j}^{a_k}$ for all L' . Hence, induced preferences satisfy the independence axiom. Continuity is direct since expected utility is continuous in probabilities.

(\Leftarrow) STEP 1. *One of the degenerated lotteries is the worst among all lotteries, another one is the best one.*

Since \succsim is a preference relation over \mathcal{L} , degenerated lotteries can be ordered from worst to best. Let \underline{L} be the worst (or one of the worst) degenerated lottery which assigns probability 1 to, say, a_1 and \bar{L} be the best (or one of the best) degenerated lottery which assigns probability 1 to, say, a_n .

All degenerated lotteries $L \succsim \underline{L}$ where a_i is assigned probability 1 can be reached as $L = \underline{L}(1\Delta_{a_1}^{a_i})$; thus

$$1\Delta_{a_1}^{a_i} \succ^{\Delta} 1\Delta_{a_1}^{a_1} \quad (1)$$

All lotteries $L = (p_1, \dots, p_n)$ in \mathcal{L} can be reached from a sequence of lotteries $L_1 = \underline{L}$, $L_2 = L_1(p_2\Delta_{a_1}^{a_2})$, ..., $L_n = L_{n-1}(p_n\Delta_{a_1}^{a_n}) = L$ and by (1) and independence, at all iterations i , $L_{i+1} \succ L_i$; thus $L \succ \underline{L}$ by transitivity of the preferences. Hence \underline{L} is the worst (or one of the worst) lottery in \mathcal{L} .

A similar argument establishes that \bar{L} is the best (or one of the best) lottery in \mathcal{L} .

STEP 2. If $\beta\Delta_{a_i}^{a_j} \succ^{\Delta} \beta\Delta_{a_i}^{a_i}$ for some L' and β , then for all $\alpha, \alpha' \in [0, p_i]$, $\alpha\Delta_{a_i}^{a_j} \succ \alpha'\Delta_{a_i}^{a_j}$ if and only if $\alpha > \alpha'$ for all L .

Without loss of generality, let $\alpha > \alpha'$, i.e.,

$$L(\alpha\Delta_{a_i}^{a_j}) = L^1((\alpha - \alpha')\Delta_{a_i}^{a_j}) \text{ where } L^1 \equiv L(\alpha'\Delta_{a_i}^{a_j}).$$

Suppose $\beta\Delta_{a_i}^{a_j} \succ^{\Delta} \beta\Delta_{a_i}^{a_i}$ for some L' and β , by independence for any L we have

$$L(\alpha\Delta_{a_i}^{a_j}) = L^1((\alpha - \alpha')\Delta_{a_i}^{a_j}) \succ L^1((\alpha - \alpha')\Delta_{a_i}^{a_i}) = L(\alpha'\Delta_{a_i}^{a_j})$$

$$\text{i.e., } \alpha\Delta_{a_i}^{a_j} \succ^{\Delta} \alpha'\Delta_{a_i}^{a_j}$$

With $\alpha \leq \alpha'$, the argument proves the contrapositive statement.

STEP 3. For all $L \in \mathcal{L}$ there exists $\alpha \in [0, 1]$ such that $\underline{L}(\alpha\Delta_{a_1}^{a_n}) \sim L$.

If $L \sim \underline{L}$ then $\alpha = 0$ and if $L \sim \bar{L}$, $\alpha = 1$.

We consider now that $\bar{L} \succ L \succ \underline{L}$. Suppose there is no such α for L . By Step 2, we know that the strictly larger α is, the strictly

better $\underline{L}(\alpha\Delta_{a_1}^{a_n})$ is. Moreover $\overline{L} = \underline{L}(1\Delta_{a_1}^{a_n}) \succ L \succ \underline{L} = \underline{L}(0\Delta_{a_1}^{a_n})$, thus:

\Rightarrow either there is one $\alpha' \in [0, 1]$ such that $\underline{L}(\alpha\Delta_{a_1}^{a_n}) \prec L$ for all $0 \leq \alpha \leq \alpha'$ and $\underline{L}(\alpha\Delta_{a_1}^{a_n}) \succ L$ for all $1 \geq \alpha > \alpha'$; continuity would not hold in this case since, $\{\alpha \in [0, 1] : \underline{L}(\alpha\Delta_a^b) \succ L\} =]\alpha', 1]$ is open;
 \Rightarrow or there is one $\alpha' \in [0, 1]$ such that $\underline{L}(\alpha\Delta_{a_1}^{a_n}) \prec L$ for all $0 \leq \alpha < \alpha'$ and $\underline{L}(\alpha\Delta_{a_1}^{a_n}) \succ L$ for all $1 \geq \alpha \geq \alpha'$; in this case continuity would not hold since, $\{\alpha \in [0, 1] : L \succ \underline{L}(\alpha\Delta_a^b)\} = [0, \alpha'[$ is open. ■